THE NINETEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE nineteenth summer meeting of the Society was held at the University of Pennsylvania on Tuesday and Wednesday, September 10–11, 1912, extending through two sessions on Tuesday and a morning session on Wednesday. The following twenty-nine members were in attendance:

Professor Frederick Anderegg, Professor M. J. Babb, Mr. A. A. Bennett, Dr. Elizabeth R. Bennett, Professor G. G. Chambers, Dr. A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor E. S. Crawley, Professor G. H. Cresse, Professor E. W. Davis, Professor W. P. Durfee, Professor H. B. Evans, Mr. Meyer Gaba, Professor O. E. Glenn, Professor G. H. Hallett, Dr. S. Lefschetz, Professor J. A. Miller, Dr. H. H. Mitchell, Mrs. Anna J. Pell, Professor M. B. Porter, Professor Arthur Ranum, Professor E. D. Roe, Jr., Dr. J. E. Rowe, Professor F. H. Safford, Professor I. J. Schwatt, Professor H. D. Thompson, Professor Oswald Veblen, Professor H. S. White.

Ex-President H. S. White occupied the chair, being relieved by Professors Crawley and Davis. The Council announced the election of the following persons to membership in the Society: Professor W. A. Bratton, Whitman College; Professor Florence P. Lewis, Goucher College; Mr. Leslie MacDill, Indiana University; Professor H. W. March, University of Wisconsin; Mr. M. R. Richardson, University of Chicago; Dr. J. I. Tracey, Johns Hopkins University; Mr. H. S. Vandiver, Philadelphia, Pa. Five applications for membership in the Society were received.

On both days of the meeting luncheon was provided by the University of Pennsylvania. On Tuesday evening twenty-six of the members gathered at the usual dinner. In the interval between the sessions the visitors were conducted through the university buildings and grounds. On Wednesday afternoon an automobile excursion about the city was arranged. At the close of the meeting a vote of thanks was tendered to the university authorities for their generous hospitality.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On the theory of relativity: analysis of the postulates."

(2) Professor F. H. SAFFORD: "An irrational transformation of the Weierstrass $\wp$-function curves" (preliminary communication).

(3) Dr. E. L. DODD: "The least square method grounded with the aid of an orthogonal transformation."

(4) Dr. E. L. DODD: "The probability of the arithmetic mean compared with that of certain other functions of the measurements."

(5) Dr. HENRY BLUMBERG: "Algebraic properties of linear homogeneous differential expressions."

(6) Dr. J. E. ROWE: "The relation between tangents and osculant $(n-1)$ics of rational plane curves."

(7) Dr. H. H. MITCHELL: "Determination of all primitive collineation groups in n (>4) variables which contain homologies."

(8) Professor ARTHUR RANUM: "Lobachevskian polygons trigonometrically equivalent to the triangle."

(9) Professor G. A. MILLER: "A few theorems relating to Sylow subgroups."

(10) Mrs. ANNA J. PELL: "Linear equations in infinitely many unknowns."

(11) Mr. L. B. ROBINSON: "Invariants of two tetrahedra."

(12) Professor F. R. SHARPE: "The Klein-Ciani quartic."

(13) Professor F. R. SHARPE: "The (2, 1) ternary correspondence with a sextic curve of branch points."

(14) Professor F. R. SHARPE and Dr. F. M. MORGAN: "A type of quartic surface invariant under a non-linear transformation of period 3."

(15) Dr. S. LEFSCHETZ: "Double curves of surfaces projected from S_4."

(16) Dr. HENRY BLUMBERG: "Sets of postulates for the rational, the real, and the complex numbers."

(17) Professor OSWALD VEBLEN: "Decomposition of an n-space by a polyhedron."

(18) Professor F. N. COLE: "The triad systems of thirteen letters."

(19) Professor H. S. WHITE: "Triple systems as transformations, and their paths among triads."

(20) Professor L. C. KARPINSKI: "Augrim stones."

(21) Dr. DUNHAM JACKSON: "On the approximate representation of an indefinite integral."

(22) Dr. T. H. GRONWALL: "Some special boundary problems in the theory of harmonic functions."

(23) Dr. T. H. GRONWALL: "On analytic functions of constant modulus on a given contour."

(24) Dr. T. H. GRONWALL: "On series of spherical harmonics."

(25) Professor O. E. GLENN: "A general theorem on upper and lower limits for the order of a factor of a p-ary form with polynomial coefficients."

(26) Professor E. J. WILCZYNSKI: "On a certain class of self-projective surfaces."

Dr. Blumberg was introduced by the Secretary, Mr. Robinson by Dr. Cohen. In the absence of the authors the papers of Professor Carmichael, Dr. Dodd, Professor Miller, Professor Sharpe, Dr. Morgan, Professor Karpinski, Dr. Jackson, Dr. Gronwall, and Professor Wilczynski were read by title.

The papers of Professor Miller and Dr. Lefschetz appear in full in the present issue of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the present paper Professor Carmichael subjects the postulates of relativity to a fresh analysis. He points out that a part of the second postulate is a consequence of the first, and calls attention to the fact that most workers in the theory of relativity have (consciously or unconsciously) made other fundamental hypotheses in addition to the two so-called postulates of relativity. These additional hypotheses should be stated as postulates; and in this paper for these additional hypotheses two very simple ones are chosen. These two together with the first and a special form of the so-called second postulate of relativity form the basis of the paper. With this as a foundation the essential general results of the theory of relativity are built up in a very elementary manner and certain extensions of previous results are obtained. Special attention is given to an analysis of those elements in the postulates which give rise to the strange conclusions of the theory. The problem of logical equivalents of the postulates is also treated.

2. In two previous papers Professor Safford has considered an irrational transformation due to Weierstrass, in which the curves obtained are of the sixteenth degree. In the case analogous to the Legendrian form of the elliptic element he

has resolved these curves into four curves of the fourth degree. But in the case corresponding to the Weierstrass form two curves are of the fourth degree, while there remains an apparently irresolvable component.

This paper is a study of the conditions under which further resolution is possible.

3. Dr. Dodd establishes the least square method, for the case in which the observation equations are linear, as a consequence of the Gaussian probability law, without recourse to infinite series, approximations, or a discontinuity factor.* Only those values for the unknowns are considered which can be obtained from linear combinations of the observation equations. It would be futile to consider all possible values for the unknowns in this connection; for under the Gaussian law, there are no "most probable values" for the unknowns.

Let x be the true value of one of the unknowns and ξ an approximation for x, obtained from a linear combination of the observation equations, with multipliers X_i. Theorem: The error of each measurement being subject to the Gaussian law, the probability that the error $x - \xi$ of ξ will lie in any given interval $(-\alpha, \alpha)$ is greatest when the X's are so chosen that ξ has the value given it by the least square method.

There being n observation equations, the required probability is expressed as an n-fold integral. This is then simplified by an orthogonal transformation—"rotation"—making the two "parallel planes" which bound the region of integration "perpendicular" to an "axis of coordinates." If the measures of precision are different, a similitude transformation is first used.

4. In his second paper Dr. Dodd investigates the consequences of the Gaussian probability law, and shows that this law is not compatible† with the principle of the arithmetic mean as the "most probable value." By the Gaussian law (G. L.) is meant the statement that

$$\frac{h}{\sqrt{\pi}} \int_{x'}^{x''} e^{-h^2x^2} dx$$

* See Czuber, Theorie der Beobachtungsfehler, p. 232.

† Bertrand, in his Calcul des Probabilités (1889), pp. 177–180, gives an example to exhibit this incompatibility.

is the probability that the error $x = a - m$ of the measurement m will lie between x' and x''; here a is the true value. Under the G. L. there is, indeed, "no most probable value" for the unknown, the probability of each of the infinite number of possible values being zero.

Definition: The probability of a function F of the measurements will be said to be greater than that of another function f if the probability that F will differ from the true value a by less than α is greater than the probability that f will differ from a by less than α, for all positive values of α less than some α'.

Then, under the G. L., the measurements having the same "measure of precision" h, the probability of the root-mean-square, $\sqrt{\frac{1}{2}(m_1^2 + m_2^2)}$, of two measurements m_1, m_2 is greater than that of the arithmetic mean $M = \frac{1}{2}(m_1 + m_2)$ if $ha > 2$. But in the case of three measurements the probability of the arithmetic mean is the greater.

Let M be the arithmetic mean of any number of measurements, with the same h. There exist values of the constant b, a little less than unity, such that the probability of bM is greater than that of M. But if $b > 1$, the probability of bM is less than that of M.

Under the G. L., the probability of the geometric mean, and of the median and of $M + c$,—this c being a constant, not zero,—is less than the probability of M.

If the measures of precision are $h_1, h_2, \cdots, h_n$, respectively, the probability of the weighted mean $W = p_1m_1 + p_2m_2 + \cdots + p_nm_n$ (where $p_1 + p_2 + \cdots + p_n = 1$) is greatest when the weights are given their usual values $p_i = h_i^2/\Sigma h^2$; but values of b exist making the probability of bW greater than that of W.

5. In his first paper Dr. Blumberg sketches the main results in his dissertation, entitled "Ueber algebraische Eigenschaften von linearen homogenen Differentialausdrücken" (Göttingen, 1912). The dissertation deals with algebraic properties (such as those connected with reducibility, irreducibility, etc.) of expressions of the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y,$$

where the functions $a_0(x)$, $a_1(x)$, . . . belong to a fixed domain of rationality, in the sense of the Picard-Vessiot group theory of linear homogeneous differential equations. The chief point to be emphasized is that the proofs make no use of integrals, but are given with the aid of a very small number of properties of linear homogeneous expressions, so that the results are of a much more general character than those hitherto obtained.

6. A rational plane curve of order n, an R^n, possesses $2n-2$ tangents through every point of the plane; also, there are $2n-2$ osculant $(n-1)$-ics of the R^n through every point of the plane. Dr. Rowe's paper consists in proving that any projective property imposed upon the parameters of the $2n-2$ osculant $(n-1)$-ics of the R^n through a point holds for the tangents through the point as a pencil of lines.

7. In a recent paper (*Proceedings of the London Mathematical Society*, November, 1911) Burnside made a determination of all finite collineation groups in n variables with rational coefficients which contain the symmetric group on those variables. In addition to the already known groups, those of order $(n+1)!$ (for any n) and $2^7 \cdot 3^4 \cdot 5$ (for $n = 6$), he found two additional primitive groups, of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ (for $n = 7$) and $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ (for $n = 8$).

Dr. Mitchell solves a more general problem, i. e., the determination of all primitive groups in $n(>4)$ variables which contain homologies, the results for $n \leqq 4$ being already known. A transformation in the symmetric group which interchanges two of the variables and leaves the rest unaltered is evidently an homology of period 2. In addition to the groups mentioned, there are two others, of order $2^6 \cdot 3^4 \cdot 5$ (for $n = 5$) and $2^8 \cdot 3^6 \cdot 5 \cdot 7$ (for $n = 6$). Neither of these can be represented with rational coefficients. The former is isomorphic with the known simple group of that order, and the latter contains a self-conjugate subgroup of index 2, which is (1, 1) isomorphic with a known simple group.

8. Professor Ranum shows that in the Lobachevskian plane there are nine polygons whose trigonometry is equivalent to that of the triangle and three special polygons whose trigonometry is equivalent to that of the right triangle. Among the former is the rectangular hexagon (whose angles are all

right angles). Among the latter is the rectangular pentagon, which bears somewhat the same relation to the right triangle in the Lobachevskian plane that Gauss's "Pentagramma Mirificum" bears to the right triangle in the Riemannian plane. The hyperbolic cosine of any side of the rectangular pentagon is equal to the product of the hyperbolic sines of the two opposite sides and to the product of the hyperbolic cotangents of the two adjacent sides.

10. A matrix (k_{pq}) is unlimited if the transformation $\Sigma_q k_{pq} x_q$ does not transform every system of variables $\{x_p\}$ of finite norm into a system of finite norm. Linear differential and differential-integral equations, and other types of linear equations, correspond directly to linear equations in infinitely many unknowns for which the matrix of the coefficients is unlimited. In Mrs. Pell's paper conditions are given for the existence of solutions of systems of homogeneous and non-homogeneous linear equations with an unlimited matrix of coefficients.

11. In this paper Dr. Robinson obtains a complete set of invariants of two tetrahedra, the one taken in points, the other in planes. The tetrahedra are written $(a\xi)(b\xi)(c\xi)(d\xi) = 0$ and $(\alpha x)(\beta x)(\gamma x)(\delta x) = 0$. An invariant is defined as a rational integral function of the coefficients homogeneous in the Greek and Roman letters and unaltered by any permutation of the letters and the subscripts. The group of permutations is of order 576. A rather full discussion is given of this group and the results obtained are compared with those obtained by Feder in a paper in the *Mathematische Annalen*, volume 47.

12. The Klein quartic (*Mathematische Annalen*, 1879)

$$x^3z + y^3x + z^3y = 0 \tag{1}$$

and the Ciani quartic (*Palermo Rendiconti*, 1899)

$$\Sigma x^4 - 3\lambda \Sigma y^2 z^2 = 0 \tag{2}$$

(for a special value of λ) are both invariant under 21 harmonic homologies. Ciani states that (2) must therefore be trans-

formable into (1) by some opportune transformation. In Professor Sharpe's paper the requisite transformation is determined and is applied to prove the following theorems. There are 63 conics that pass through one vertex of each of the 8 inflectional triangles, 21 of which pass through each vertex. There are 168 conics that pass through 2 vertices of each of 4 of the 8 inflectional triangles, 56 of which pass through each vertex. There are 28 conics that pass through the 6 vertices of each pair of inflectional triangles and through the points of contact of an associated bitangent.

13. Noether (*Mathematische Annalen*, 1889), reduced the double planes that can be rationally mapped on simple planes to three types according as the curve of branch points is (1) a C_{2n} with a $(2n-2)$-fold point, (2) a non-singular quartic, (3) a sextic having 3 branches touching the same line at a common point. A complete analytical treatment of the second type was given by De Paoli (*Atti Lincei*, 1878), and of the first type by Boyd (*American Journal*, 1912). In Professor Sharpe's second paper precise analytical expressions are determined for the remaining type of (2, 1) correspondence by means of the Grassmann and Wiman (*Mathematische Annalen*, 1895) depictions of a cubic surface on a simple and double plane.

14. Professor Sharpe and Dr. Morgan consider a quartic surface whose section by any plane through a line AB is the line and a cubic through A and B. If the line joining A to any point P on the surface meets the surface in Q and if BQ meets the surface in R, the transformation used converts P into R. In the most general case the conditions for periodicity give 17 relations amongst the 26 coefficients of the equation of the surface. When certain terms are absent from the equation the conditions for periodicity are simple.

16. In his second paper Dr. Blumberg deals with sets of postulates for rational, real, and complex numbers. These sets grew out of conversations with Professor Zermelo and it is the intention of the latter and Dr. Blumberg to publish their results in detail in the near future. Starting with a set (F) of postulates defining the most general field (Körper), where the commutative law of multiplication need not hold,—such a field

will be henceforth simply called a "non-commutative field"—we obtain by the addition of a postulate R a set of postulates defining the most general non-commutative field that contains as a sub-field the field of rational numbers. By the further addition of one postulate C, we obtain a set of postulates defining the most general non-commutative field that contains as a sub-field the field of real numbers. Finally, the set of postulates (F), R, C and a new postulate I define the most general non-commutative field containing as a sub-field the field of complex numbers. To obtain sets of postulates defining the rational, real, and complex numbers a postulate P is added respectively to the second, third, and fourth set of postulates mentioned above, as shown below:

For the rational numbers: (F) R, P.
For the real numbers: (F), R, C, P.
For the complex numbers: (F), R, C, I, P.

There are no postulates of order. In the proofs the concept of finite number is not presupposed.

17. Professor Veblen's paper gave a proof of the theorem that an $(n-1)$-dimensional polyhedron decomposes an n-dimensional space into two regions. The proof is of an essentially combinatorial character and involves few geometrical ideas beyond the principle that an n-dimensional convex region is decomposed into two n-dimensional convex regions by an $(n-1)$-space containing one of its points.

18. Triad systems in thirteen letters were given by Kirkman, Reiss, Netto, and De Vries. But the complete determination of these systems, of which there are only two distinct types, was first effected by De Pasquale* and Brunel,† both of whom based their examination on the configurations of S. Kantor. Professor Cole determines the possible systems by direct construction without the use of any special apparatus.

In the discussion of the paper, Professor Veblen pointed out that the two systems might be obtained as an application of finite geometry.

19. In a triple system every pair of elements (or dyad)

* "Sui sistemi ternari di 13 elementi," *Rendiconti R. Istituto Lombardo*, ser. 2, vol. 32 (1899).

† "Sur les deux systèmes de triades de treize éléments," *Journal de Math.*, ser. 5, vol. 7 (1901).

occurs in some triad once only, and the system may therefore be regarded as relating every dyad to some one element, the one which occurs with it in a triad. By this means every triad, as containing three dyads, may be converted into another triad or set of three elements. Iteration produces trains of triads, terminating in recurrent cycles. For the two triple systems on 13 elements these trains are studied. They are found to be characteristic for the species, and incidentally to facilitate the discovery of the group which transforms the triple system into itself.

20. Professor Karpinski explains the system of calculation in use from the time of Gerbert (c. 1000 A.D.) through several centuries, employing Hindu numerals upon an abacus. The numerals were placed upon markers (or stones or apices) and used instead of the corresponding set of stones. This accounts for the self-contradictory expression *augrim* (or algorism) stones, as found in Chaucer's Canterbury Tales. The word algorism was widely used to indicate the Hindu art of reckoning, employing the zero, whereas the term stones implies the use of an abacus. This article is to appear in *Modern Language Notes*.

21. From a theorem which he recently reported to the Society, concerning the approximate representation of a periodic function satisfying a Lipschitz condition, Dr. Jackson deduces the following general result: If $f(x)$ is a function of period 2π which can be approximately represented for all values of x by a trigonometric sum of the nth order with an error not exceeding ϵ, and if $f(x)$ is such that its indefinite integral also is periodic, then this indefinite integral can be represented by a trigonometric sum of the nth order or lower with an error not exceeding $6\,\epsilon/n$. In a case of particular interest, the factor 6 may be replaced by 3. Of the consequences of this theorem, the following may be mentioned: If $f(x)$ has the period 2π and possesses a $(k-1)$th derivative which satisfies a Lipschitz condition with coefficient λ, then $f(x)$ can be approximately represented by a trigonometric sum of the nth order or lower with a maximum error not greater than $3^k\lambda/n^k$. This is an improvement over the result previously established in this connection. Corresponding but not precisely parallel theorems may be stated concerning approximation by means of polynomials.

22. Dr. Gronwall shows how the theory of the Cesàro summability of the Fourier and Laplace series, coupled with the generalizations of Abel's continuity theorem for power series due to Frobenius and Hölder, gives a simple and effective method for solving boundary problems in mathematical physics in the case of a circular ring or a hollow sphere. The method is illustrated, first by Dirichlet's problem for a circular ring (previously solved by Villat) and a hollow sphere, in which cases the proofs are considerably shortened, and second, by the determination of a harmonic function in a circle, where the derivative along the normal is given on part of the circumference, the function itself being given on the remaining part; in this case (also treated by Villat) not only the proof is shorter, but the final formula is considerably simplified.

23. In this paper, Dr. Gronwall considers a uniform analytic function $f(x)$ having a finite number of essential singularities inside or on a given circle, and being of constant modulus on a set of points on the circumference with at least one limit point which is not an essential singularity. After showing that, in consequence of the last condition, the modulus is constant in every regular point on the circumference, the singularities and zeros outside the circle are determined by Schwarz' method of analytic continuation, and finally the general analytical expression of the function is obtained.

24. In the first part of this paper, Dr. Gronwall considers the $(n+1)$th partial sum in the formal development of a function $f(\theta, \varphi)$ in spherical harmonics according to Laplace's formula. When $f(\theta, \varphi)$ is limited to the class of continuous functions not exceeding unity in absolute value at any point of the sphere, this sum has an upper limit ρ_n, and it is shown that

$$\lim_{n=\infty} \frac{1}{\sqrt{n}} \cdot \rho_n = 2\sqrt{\frac{2}{\pi}},$$

which completes the result previously found by Haar

$$\liminf_{n=\infty} \frac{1}{\log n} \cdot \rho_n > 0.$$

As applications, simple examples of everywhere continuous

functions are formed, the Laplace developments of which are divergent at a given point, or non-uniformly convergent.

The second part of the paper is concerned with the summability of Laplace's series by the Cesàro (or Hölder) means. Fejér has established the summability of the second order for any absolutely integrable function in every point where it is continuous; Haar, considering the particular case of Legendre's series for any function $f(x)$ (where $x = \cos\theta$), continuous for $-1 \leqq x \leqq 1$, proved the summability of the first order except at the end points -1 and $+1$, which are inaccessible to his method and were treated by Chapman under additional restrictions regarding $f(x)$.

Dr. Gronwall defines a set of constants ρ_n' bearing the same relation to the arithmetic mean of the $n+1$ first partial sums of Laplace's series as ρ_n to the $(n+1)$th sum itself, and shows that these constants ρ_n' are bounded for all values of n. From this result, the following general theorem is derived: The Laplace series of any absolutely integrable function $f(\theta, \varphi)$ is summable of the first order at every point where the function is continuous, and this summability is uniform over any range interior to a continuity range.

Finally, it is shown that when $f(\theta, \varphi)$ is integrable, without being absolutely integrable, the corresponding Laplace series is summable of the second order at every point of continuity.

25. Professor Glenn takes a homogeneous p-ary form written in the ordinary manner, without multinomial multipliers with its coefficients, and considers its coefficients to be polynomials in one letter x, after the method of Königsberger* in his generalizations in the binary case of Eisenstein's theorem. A general range of functions is then considered, consisting of k of these polynomials arranged in a certain order. Derived ranges are defined. A formula for the number of functions on a general range is given. Certain assumptions are then made with reference to the existence of common factors of the functions on a general range, and of the functions on its derived ranges and it is proved that the order n of any factor of the p-ary form must, under the given hypothesis lie between limits, which are determined. The theorem is applied in proving certain general families of forms irreducible.

* *Journal für Math.*, vol. 115.

26. Several years ago Professor Wilczynski showed that, by introducing a properly chosen system of projective coordinates, the equation of a non-ruled surface, in the vicinity of an ordinary point, may be replaced by a development of the form

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \ldots,$$

where I, J and all higher coefficients of this expansion are absolute differential invariants of the surface. The present paper is devoted to an investigation of those special surfaces for which I and J are everywhere equal to zero, completely determines these surfaces in certain elementary cases, and obtains a large number of properties for them.

F. N. COLE,
Secretary.

A FEW THEOREMS RELATING TO SYLOW SUBGROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 10, 1912.)

SUPPOSE that a group G involves more than one Sylow subgroup of order p^m, and that a subgroup H of G involves more than one Sylow subgroup of order p^β, $0 < \beta < m$. The number of the subgroups of order p^β in H cannot exceed the number of those of order p^m in G, since any two Sylow subgroups of any group generate a group whose order is divisible by at least two distinct prime numbers, and hence each Sylow subgroup of order p^β in H occurs in one and in only one Sylow subgroup of order p^m in G.

When H is an invariant subgroup of G it is easy to prove that the number of the Sylow subgroups of G is a multiple of the number of the corresponding Sylow subgroups of H. In fact, all the operators of G which transform a subgroup of order p^m into itself must also transform into itself all the operators of the subgroup of order p^β in H which is contained in the particular subgroup of order p^m under consideration. Hence it results that all the operators of G which transform into itself a subgroup of order p^β contained in H must constitute a group involving k subgroups of order p^m and containing

all the operators of G which transform any of these k subgroups into any other one of them.

From this it follows that every operator of G which is not in the subgroup composed of all the operators of G which transforms into itself one of the given subgroups of order p^β must transform the k subgroups of order p^m which transform this subgroup of order p^β into itself into other k subgroups such that the two sets of k subgroups have none of these subgroups in common. In other words, there are exactly k times as many subgroups of order p^m in G as there are subgroups of order p^β in H. It is clear from the above that the group of transformations of the Sylow subgroups of order p^m in G must be imprimitive whenever $k > 1$. When H is non-invariant it is not always true that the number of its subgroups of order p^β is a divisor of the number of the subgroups of order p^m in G. Hence the theorem:

If a group G, involving Sylow subgroups of order p^m, contains an invariant subgroup H which involves Sylow subgroups of order p^β, then the number of the subgroups of order p^β in H is always a divisor of the number of the subgroups of order p^m in G; when the former of these two numbers is larger than the latter, G transforms its subgroups of order p^m according to an imprimitive group.

If G is the symmetric group of degree n and H is the alternating group of the same degree, it is clear that the Sylow subgroups of p^m in G are the same as those of H, except when $p = 2$. In this special case the order of the Sylow subgroups of G is twice the order of the corresponding Sylow subgroups of H. When $n = 4$, it is well known that H contains only one Sylow subgroup of order 4 while G contains three Sylow subgroups of order 8. When $n = 5$, H contains five Sylow subgroups of order 4 while G contains fifteen Sylow subgroups of order 8. When n is less than 4, H does not involve any Sylow subgroups of order 2^m. We proceed to prove that in all other cases the number of the Sylow subgroups of order 2^m in the symmetric group of degree n is the same as the number of the Sylow subgroups of order 2^{m-1} in the corresponding alternating group. It is easy to verify that this theorem is true when $n = 6$, and hence we may employ the method of complete induction in proving the general theorem.

We may first observe that the number of the Sylow subgroups of order p^m in every transitive group of degree p^n is

exactly the same as the number of the Sylow subgroups of order $p^{m-\alpha}$ in the subgroup G_1 composed of all the substitutions which omit one letter of the given transitive group G of degree n whenever $\alpha < m$. This theorem follows directly from the facts that the number of the Sylow subgroups of G cannot be less than the number of the corresponding Sylow subgroups in a subgroup of G, and that all the subgroups of order p^m in G which involve a subgroup of order $p^{m-\alpha}$ contained in G_1 are conjugate under G_1. Each of these subgroups of order p^m involves only one of the given Sylow subgroup of order $p^{m-\alpha}$, and is transformed into itself under G by a group whose order is p^α times the group which transforms it into itself under G_1. As the order of G is also p^α times the order of G_1, it results that each of the Sylow subgroups of order p^m in G involves one and only one Sylow subgroup of order $p^{m-\alpha}$ in G_1. In other words, the number of Sylow subgroups of order $p^{m-\alpha}$ in G_1 is the same as the number of the Sylow subgroups of order p^m in G.

From the preceding paragraph it results that if a Sylow group of order 2^m is transformed into itself by only its own substitutions in the alternating and the symmetric group of degree $2^\alpha - 1$, it is transformed into itself by only its own substitutions in the alternating and the symmetric group of degree 2^α_{+1}. The Sylow subgroup of order 2^m in a symmetric group, or in an alternating group, is transitive only when the degree of this group is either 2^α or $2^\alpha + 1$*. From the structure of this group it follows therefore that the Sylow subgroups whose orders are of the form 2^m are transformed into themselves only by their own substitutions in every symmetric group and in every alternating group whose degree exceeds 5. Hence the theorem:

The number of the Sylow subgroups of order 2^m in the symmetric group of degree $n > 5$, is exactly the same as the number of the Sylow subgroups of order 2^{m-1} in the alternating group of this degree, and these Sylow subgroups are transformed into themselves by only their own substitutions under each of these groups.

From this theorem it results that the group of order 4 is the only Sylow subgroup, whose order is of the form 2^m, which is transformed into itself by more than its own operators under

* *Amer. Jour. of Mathematics*, vol. 23 (1901), p. 176.

a symmetric or under an alternating group. It may also be observed that if G is an imprimitive group of transformations of Sylow subgroups of order p^m, then G cannot involve an invariant subgroup whose order is of the form p^α, since its degree is of the form $1 + kp$. If the systems of imprimitivity of G are transformed according to a group involving smaller Sylow subgroups of the form p^α than those contained in G, it results from the theorem proved above that G contains other systems of imprimitivity which are transformed according to Sylow subgroups whose orders of the form p^α are equal to the orders of the corresponding Sylow subgroups of G. Hence the theorem:

If G is an imprimitive group of transformations of Sylow subgroups of order p^m and involves Sylow subgroups of order p^α, then G must have systems of imprimitivity which are transformed according to a group involving Sylow subgroups of order p^α.

UNIVERSITY OF CHICAGO.

THEOREMS ON FUNCTIONAL EQUATIONS.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, April 27, 1912.)

1. IN the BULLETIN, volume 18 (1912), page 300, we have referred to Abel, *Crelle's Journal*, volume 2 (1827), page 389, in relation to the equation

$$\psi(x) - \psi(y) = \Omega^{-1}\{\phi(x, y)\}. \tag{1}$$

This reference suggests

$$\psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\} \tag{2}$$

as a correlative of the functional equation* discussed by Abel, l. c., namely,

$$\psi(x) + \psi(y) = \Omega^{-1}\{x\phi(y) + y\phi(x)\}. \tag{2'}$$

Further special cases of the equation (1) are obtained by considering the generalizations of equation (2') by Lottner,

* Cf. Cayley, Mathematical Papers, vol. IV, pp. 5–6.

Crelle's Journal, volume 46 (1854), pages 367, 368, etc. For example, we obtain

$$\psi(x) - \psi(y) = \Omega^{-1}\left\{\frac{x\phi(y) - y\phi(x)}{\theta(xy)}\right\} \tag{3}$$

and

$$\psi(x) - \psi(y) = \Omega^{-1}\left\{\frac{\Phi(x)\phi(y) - \Phi(y)\phi(x)}{\theta(xy)}\right\}. \tag{4}$$

In equations (2), (3), (4) let $\Omega^{-1}(x) = \psi(x)$; then the resulting equations are particular instances of the equation

$$\chi(x) - \chi(y) = \chi\{f(x, y)\},$$

which we have derived in the BULLETIN, l. c. Following the suggestions of Abel, l. c., page 386, and Lottner, l. c., page 368, we remark that equation (2) for $\Omega^{-1}(x) = \psi(x)$ contains a trigonometric subtraction formula, and equation (3) for $\Omega^{-1}(x) = \psi(x)$ contains an elliptic subtraction formula.

Abel, l. c., pages 388–389, has pointed out that the relation (2′) leads to an equation in $\phi(x)$ which in general is incapable of solution. The relation (2), on the contrary, gives

$$\begin{aligned} \phi(x) &= \sqrt{c^2 - (m - \alpha'^2)x^2} + \alpha' x, \\ \psi(x) &= a\alpha \int \frac{dx}{\sqrt{c^2 - (m - \alpha'^2)x^2}}, \\ \Omega^{-1}(x) &= \psi\left(\frac{x}{\alpha}\right) + \psi(0), \end{aligned} \tag{5}$$

where $\phi'(0) = \alpha'$, $\phi(0) = \alpha$, $\psi'(0) = a$, and c and m are arbitrary constants.

In connection with the discussion of Lottner, l. c., and the preceding equation (4) the functional equation of Lelieuvre* is possibly of interest.

2. Abel, in *Crelle's Journal*, volume 1 (1826), pages 11–15, has shown that if

$$f(x, y) = \phi^{-1}\{\phi(x) + \phi(y)\},$$

then

$$f\{x, f(y, z)\} = f\{y, f(x, z)\},$$

* *Bulletin des Sciences Mathématiques*, ser. (2), vol. 27 (1903), p. 31.

and in the BULLETIN, l. c., page 302, we have shown that if

$$f(x, y) = \phi^{-1}\{\phi(x) - \phi(y)\},$$

then

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}.$$

The following theorems have a bearing on these two remarks.

THEOREM 1. *If* $\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}$ *and* $\phi\{f(y, z), z\} = y$ *and* $f\{\phi(y, z), y\} = z$, *then there exists a function* $\chi_\phi(x)$ *such that* $\phi(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) + \chi_\phi(y)\}$ *and* $f(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) - \chi_\phi(y)\}$.

In proof of this theorem we show first that

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(x, y)\}.$$

By hypothesis,

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}. \tag{6}$$

Hence writing $f(y, z)$ for y, we get

$$\phi\{x, \phi[f(y, z), z]\} = \phi\{f(y, z), \phi(x, z)\},$$

or since

$$\phi\{f(y, z), z\} = y$$

we obtain

$$\phi(x, y) = \phi\{f(y, z), \phi(x, z)\},$$

or by interchanging y and z,

$$\phi(x, z) = \phi\{f(z, y), \phi(x, y)\}.$$

In this relation we write $\phi(y, z)$ for z; then

$$\phi\{x, \phi(y, z)\} = \phi\{f[\phi(y, z), y], \phi(x, y)\},$$

that is, since

$$f\{\phi(y, z), y\} = z,$$

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(x, y)\}. \tag{7}$$

Combining (6) and (7), we have by Abel's theorem, *Crelle's Journal*, volume 1, page 13, that there exists a function $\chi_\phi(x)$ such that

$$\chi_\phi\{\phi(x, y)\} = \chi_\phi(x) + \chi_\phi(y). \tag{8}$$

Therefore,

$$\chi_\phi\{\phi[f(x, y), y]\} = \chi_\phi\{f(x, y)\} + \chi_\phi(y)$$

or

$$\chi_\phi(x) = \chi_\phi\{f(x, y)\} + \chi_\phi(y),$$

that is,

$$\chi_\phi\{f(x, y)\} = \chi_\phi(x) - \chi_\phi(y). \tag{9}$$

The relations (8) and (9) establish the preceding theorem.

THEOREM 2. *If* $f\{y, f(x, z)\} = f\{z, f(x, y)\}$ *and* $\phi\{f(y, z), z\} = y$ *and* $f\{\phi(y, z), y\} = z$, *then there exists a function* $\chi_f(x)$ *such that* $f(x, y) = \chi_f^{-1}\{\chi_f(x) - \chi_f(y)\}$ *and* $\phi(x, y) = \chi_f^{-1}\{\chi_f(x) + \chi_f(y)\}$.

Since

$$f\{\phi(y, z), y\} = z$$

we have

$$f\{\phi[f(y, z), z], f(y, z)\} = z.$$

But

$$\phi[f(y, z), z] = y.$$

Hence,

$$f\{y, f(y, z)\} = z. \tag{10}$$

Now we consider

$$f\{y, f(x, z)\} = f\{z, f(x, y)\}. \tag{11}$$

From (11) follows

$$f\{f(x, y), f(x, z)\} = f\{z, f[x, f(x, y)]\}$$

or by (10),

$$f\{f(x, y), f(x, z)\} = f(z, y). \tag{12}$$

Hence by a theorem which we have proved, BULLETIN, l. c., there exists a function $\chi_f(x)$ such that

$$\chi_f\{f(x, y)\} = \chi_f(x) - \chi_f(y). \tag{13}$$

Therefore,

$$\chi_f\{f[\phi(x, y), x]\} = \chi_f\{\phi(x, y)\} - \chi_f(x)$$

or

$$\chi_f(y) = \chi_f\{\phi(x, y)\} - \chi_f(x),$$

that is,

$$\chi_f\{\phi(x, y)\} = \chi_f(x) + \chi_f(y). \tag{14}$$

Thus from (13) and (14) follows the proof of our second theorem.

3. Theorem 1 of § 2 concerned the *consistence* of the relations

$$\phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\}, \tag{6}$$

$$\phi\{f(y, z), z\} = y, \tag{6'}$$

$$f\{\phi(y, z), y\} = z. \tag{6''}$$

These relations, on the other hand, may be shown to be *completely independent.** To do this, it suffices to exhibit eight systems, each of which satisfies or contradicts the preceding relations. Denoting by the symbol $(+\,+\,-)$ that (6) is satisfied, (6′) is satisfied, (6″) is contradicted, and so on, we have as the required systems:

	$\phi(x, y) =$	$f(x, y) =$
$(+\,+\,+)$	$x + y$	$x - y$
$(+\,+\,-)$	$= -x + y$	$= -x + y$
$(-\,+\,+)$	$= -x - y$	$= -x - y$
$(+\,-\,+)$	$= -x + y$	$= x + y$
$(+\,-\,-)$	$= x + y$	$= x + y$
$(-\,+\,-)$	$= x - y$	$= x + y$
$(-\,-\,+)$	$= x - y$	$= -x + y$
$(-\,-\,-)$	$= x - y$	$= x - y$

CHICAGO, ILL.,
July, 1912.

DOUBLE CURVES OF SURFACES PROJECTED FROM SPACE OF FOUR DIMENSIONS.

BY DR. S. LEFSCHETZ.

(Read before the American Mathematical Society, September 10, 1912.)

1. IN the study of a surface F in S_4, one of the problems is the determination of the genus π' of the double curve of a projection of F. Severi† has shown that a general F_m in S_4 has four fundamental projective characters, viz., the order m, the class n, the rank a of a hyperplane section, and the number t of trisecants through an arbitrary point O. He also gives an expression for the rank of the double curve in question, from which, knowing the number of pinch points J of F'_m in projection of F_m, we can obtain π'. It is not uninteresting

* Cf. E. H. Moore, The New Haven Mathematical Colloquium (1910), p. 82, §47.

† "Intorno ai punti doppi impropri . . . ," *Palermo Rendiconti*, vol. 15 (1901), p. 32.

to show how π' can be obtained directly, and this is what we will do, discussing afterwards the case of a complete intersection in S_4.

2. If h is the order of the double curve, we have

$$h = \tfrac{1}{2}m(n-1) - a,$$

$$p_n = \binom{m-1}{3} - h(m-4) + \pi' + 2t - 1,$$

where p_n is the arithmetic genus of F_m. This gives

$$\pi' = h(m-4) + p_n - 2t - \binom{m-1}{3} + 1$$

$$(1) \quad = \tfrac{1}{6}(n-1)[2n^2 - 7n - 6] - \tfrac{1}{2}a(n-4) - 2t + p_n + 1.$$

Another method of obtaining π' could be used if we knew the genus π of the curve of which the double curve is the projection. For the two are in (2, 1) relation, and the coincidences occur on the curve in S_4 only and are: (1) 2 for each triple point, or $2t$; (2) one for each pinch point, or j, so that we have*

$$2t + j = 2(\pi - 1) + 4(\pi' - 1),$$

$$(2) \quad \therefore \qquad \pi' = \tfrac{1}{4}[2(\pi - t + 1) - j].$$

3. If we wish to apply either method to the case of an $F_{\mu\nu}$, complete intersection of a V_μ and a V_ν, we must calculate t, and for the first method we must find p_n. The other characters are indeed easily obtained.

$$(3) \qquad m = \mu\nu,$$

$$(4) \qquad h = \tfrac{1}{2}\mu\nu(\mu - 1)(\nu - 1),$$

$$(5) \quad \pi = \binom{h}{2} - 2h[2h - \mu - \nu - (\mu - 1)(\nu - 1) + 2],$$

$$(6) \qquad j = \mu\nu\,(\mu - 1)(\nu - 1) = 2h.$$

We obtain (5) by remarking that the curve in S_4 is the complete intersection of V_μ, V_ν and a $V_{(\mu-1)(\nu-1)}$ and applying Veronese's† formula for the number of apparent double points of a curve in n-space, while (6) is obtained by remarking that j is also the number of tangents drawn from O to $F_{\mu\nu}$, and is therefore the number of points common to V_μ, V_ν and their first polars with respect to O.

* Clebsch, Leçons de Géométrie, vol. II, p. 169.

† "Behandlung der projectivischen Verhältnisse . . . ," *Math. Ann.*, vol. 19.

4. *Value of p_n.* It is easy to show that $F_{\mu\nu}$ *is a regular surface.* In fact the system $|G'|$ adjoint to the system $|G|$ of the hyperplane sections of $F_{\mu\nu}$ is cut out by all the $V_{\mu+\nu-4}$ which do not pass through F, and they are cut by the hyperplane H of C in the system of all surfaces of order $\mu+\nu-4$ which do not contain C, surfaces which cut out on C the complete canonic series on the curve.* $|C'|$ cuts therefore on C a series of defect 0. $\therefore p_g - p_n = 0$,† which was to be proved. To find p_g we must obtain the number of conditions that a $V_{\mu+\nu-5}$ shall go through $F_{\mu\nu}$, since the canonic system of F is cut out by the $V_{\mu+\nu-5}$ that do not go through $F_{\mu\nu}$. More generally we will propose *to find the number N_l of conditions that a V_l shall go through $F_{\mu\nu}$.* For this purpose we consider the curves C_i obtained in cutting by hyperplanes H_i $(i = 1, 2, \ldots, l)$. C_i is the complete intersection of two surfaces of order μ, ν in H_i, and has for genus

$$p = \tfrac{1}{2}\mu\nu(\mu+\nu-4)+1.$$

Let r_i be the number of conditions that a V_l going through $C_1, C_2, \ldots, C_{i-1}$ has to satisfy in order to go also through C_i. Such a variety already meets C_i in $(i-1)\mu\nu$ points, contained in $i-1$ planes. The series $g^{r}_{(l-i+1)\mu\nu}$ which its variable intersections with C_i cut on the curve is complete, since it is also cut out by a complete system of adjoints, namely the surfaces of order $l-i+1=\alpha$ of H_i, which go through the fixed group of $(i-1)\mu\nu$ points. But the surfaces of order α in H_i, cut out on C_i a series of same degree, complete and contained in the preceding, therefore coincident with it. Hence if $\nu < \mu$ and

$$\alpha < \nu, \text{ then } r_i = \binom{\alpha+3}{3},$$

$$\nu \leqq \alpha < \mu, \text{ then } r_i = \binom{\alpha+3}{3} - \binom{\alpha-\nu+3}{3},$$

$$\mu \leqq \alpha \leqq \mu+\nu-4, \text{ then } \mu+\nu-4 = \alpha \leqq \nu-4,$$

and if $r_i - 1$ is the residual special series, we have by applying Riemann-Roch's theorem

$$2(r_i - r_i') = 2\alpha\mu\nu - 2(p-1);$$

$$\therefore r_i = \alpha\mu\nu - (p-1) + \binom{\mu+\nu-\alpha-1}{3},$$

* Noether, "Zur Theorie des eindeutigen Entsprechens . . . ," *Math. Ann.*, vol. 8 (1874).

† E. Picard et Simart, Traité des fonctions algébriques de deux variables, 2, pp. 437, 489.

(4) $$\mu + \nu - 4 < \alpha.$$

Then the series cut out by V_l is not special.

$$\therefore r_l = \alpha\mu\nu - p + 1.$$

It follows that

$$N_l = \sum_1^l r_i + 1.$$

In particular

$$N_{\mu+\nu-5} = \sum_{a=\mu-\nu}^{\mu-1} \binom{p+3}{3} + \sum_{a=\mu}^{\mu+\nu-5} (\alpha\mu\nu - p + 1) + \sum_{a=1}^{\nu-4} \binom{\nu-a}{3}.$$

Also

(7) $$p_g = p_n = N_{\mu+\nu-5};$$

hence (1°) If $l \geqq \mu + \nu - 5 \geqq \mu \geqq \nu$,

$$N_l = \sum_{a=\mu+\nu-4}^{l} (\alpha\mu\nu - p + 1) + 1 + (p_n - 1) + 1$$

$$= \sum_{a=1}^{l} (\alpha\mu\nu - p + 1) + p_n + 1 = \mu\nu\binom{l+1}{2} - l(p-1) + p_n + 1$$

in virtue of the expression of p.*

(2°) If $l \leqq \mu + \nu - 5$,

$$N_l = p_n - \sum_{l+1}^{\mu+\nu-5} (\alpha\mu\nu - p + 1) - \sum_{l-\mu+2}^{\nu-4} \binom{\nu-a}{3}.$$

5. *Value of t.* Let $t = \varphi(\mu, \nu)$. Applying to a V_ν formed of a V_1 and a $V_{\nu-1}$, we have

$$\varphi(\mu, \nu) = \varphi(\mu, \nu-1) + \mu\binom{\nu}{2}(\nu-1)(\nu-2) - \mu(\mu-1)(\nu-1)(\nu-2),$$

where the second term is the number of bisecants of the surface $(V_\mu, V_{\nu-1})$ meeting also the surface (V_μ, V_1), while the third is the number of these chords going through the intersection of $V_{\nu-1}$, V_μ and V_1. Taking $\sum_1^5 \varphi(\mu, i)$ and simplifying, we obtain

$$t = \varphi(\mu, \nu) = 6\binom{\mu}{3}\binom{\nu}{3}.$$

We see that (2) is easiest to apply in the case of a complete intersection, and indeed in this case comparing both values of π', we have the easiest way of obtaining p_n.

* Cf. Severi, "Su alcune questioni di postulazione," *Palermo Rendiconti*, vol. 17 (1903), p. 87.

6. The curve of S_4 of which the double curve of $F'_{\mu\nu}$ is the projection is a complete intersection, but such is not the case for the double curve itself. In this respect the case where $\mu = \nu = 3$ is instructive. For then it is found that $t = 6\binom{3}{3} = 6$, $p_g = p_a = 5$,

$$\pi = 5\cdot 18 + 5 - \binom{6}{3} - 2\cdot 6 + 1 = 28,$$
$$h = \tfrac{1}{2}3\cdot 3\cdot 2\cdot 2 = 18 = 6\cdot 3 = 9\cdot 2.$$

The complete intersection of an F_9 and an F_2 with six triple points is of genus 63, while the intersection of an F_6 and an F_3 with six triple points is of genus 28, and yet the curve with the above characteristics cannot lie on an F_3, for then it would meet F_9' in a curve of order 36. If $F_{\nu(\mu-1)(\nu-1)}$ is the intersection of V_ν and $V_{(\mu-1)(\nu-1)}$, then the curve in S_4 is on $F_{\nu(\mu-1)(\nu-1)}$, the residual intersection being of order $\mu\nu(\mu-1)(\nu-1)^2$, and its genus p_1 is found by remarking that if we project from another center this curve will be residual of one of order $\mu\nu(\mu-1)(\nu-1)$ and genus π, so that

$$\pi_1' = \pi + \tfrac{1}{2}\mu\nu(\mu-1)(\nu-1)(\nu-2)[\mu\nu + \nu(\mu-1)(\nu-1) - 4].$$

Thus, above, the residual curve is of order 72 and genus 397.

LINCOLN, NEB.,
August 20, 1912.

GEOMETRICAL OPTICS.

The Principles and Methods of Geometrical Optics, especially as Applied to the Theory of Optical Instruments. By JAMES P. C. SOUTHALL. New York, The Macmillan Company, 1910. xxiii+626 pp. with 170 figures.

THAT mathematicians and physicists have left the field of geometrical optics so largely to the scientific experts of the best firms of optical engineers may be but one of the signs of our ever increasing specialization and its accompanying narrowing of interests. Yet the association of such names as Euler, Fermat, Gauss, Hamilton, Kummer, Moebius, Sturm shows that once mathematicians contributed largely to the subject and were inspired by it; a similar state of affairs is true in regard to physicists.

From an impartial viewpoint it can hardly be gainsaid that there are at present more points of contact between geometrical

optics and mathematics than between it and physics. For although too great a forgetfulness of the physical foundation (waves) as against the mathematical formulation (rays) may sometimes be inimical to obtaining correct results in geometrical optics,* the science of the physicist bears but lightly upon that of the optician and the efforts of the optician throw but little light upon those fundamental problems of physics which at present are the most absorbing.

To the mathematician, however, there is the opportunity of applying the theory of collineations in a manner which would make an interesting and instructive addition to any course on projective or descriptive geometry, there is a chance for the theory of congruences, especially normal congruences and focal surfaces, the calculus of variations may also find an application, and finally the Lie theory of contact transformations. Nevertheless it was not a pure geometer but Maxwell first, and Abbe later, who clearly perceived the seemingly obvious fact that the elementary theory of paraxial optical imagery was fundamentally a matter of collineations pure and simple, and it was Bruns, well-known for many a theoretical application of mathematics, who made such use of Lie's work in the more advanced parts of optical theory. It is indeed a pity that in all courses on contact transformations there should not be inserted enough of the applications to mechanics and optics to enliven the students' interest in the practical availability of the subject.

As Southall is more interested in this text in such parts of geometrical optical theory as lead to results applicable to the theory of optical instruments, he can give only minimal attention to the theory of congruences, the calculus of variations, and Lie's contact transformations, and he must content himself for the most part with mere references to other sources. He does, however, take up the collinear aspect in detail (pages 162–262, passim). It is to be prayed that, despite its practical tendencies, the text will appeal to mathematicians, and were it not for the hope that we may offer some encouragement to mathematicians to include in their courses at least a slight reference to optical applications, it is doubtful if we should have the incentive to compose this notice.

* Those who are now trying to revert somewhat to a corpuscular theory of light may succeed in replacing the wave theory by a ray theory, but such a theory would still have to account for such phenomena as at present fall under the topic of waves as distinguished from rays in the sense here used.

In the matter of book-writing on an intricate subject, the author has surely set us a model. The notation is most carefully chosen and consistently followed. To aid the reader there is an appendix of thirty pages in which the notations are tabulated with their appropriate explanations. This may at first seem too much of a good thing, yet it is a feature which cannot fail to be appreciated by students of geometrical optics. There are so many points—object points, image points, primary and secondary, focal points, "Flucht" points, etc.—which can best be kept distinct in a long treatise only by distinct notational conventions. There is also a necessary profusion of significant rays and planes, of angular and linear magnitudes, and of non-geometrical objects. The figures throughout the text are both numerous and well drawn; this alone is a considerable accomplishment and must have been a difficult task, but it is a great aid in rendering the clearly written text thoroughly transparent. The analytical index is long and critically constructed. The difficult proof reading must have been performed with exceptional care, but we have observed a few errors, all trivial.* References to other works are frequent though well selected, and the historical significance of the different advances in geometrical optics is not overlooked. In short the whole text manifests an affection of the author for his subject, and the care and industry which only such an affection could inspire.

It must be encouraging to the author to see the enthusiastic reception accorded to his work by various reviewing journals. The most flattering attention, however, which a work on geometrical optics could have is a translation into German; for the chief modern optical treatises and researches are German, and only a book of the highest excellence could receive an invitation into their society. Already a translation of Southall's Geometrical Optics is in preparation by Drs.

* These are included (except for the peculiar arrow head on p. 391, line 9 from the bottom) in the list of seven furnished us by the author under the date of April 15, 1912.

P. 138. In Fig. 53 insert letter N at end of dotted line BC.
P. 236. Formula at end of page should read: $Y = Z = \pm \sqrt{-f/e'}$.
P. 250. In formula (136), read $E_2'E' = -f_2e_2'/\Delta$.
P. 278. In the "legend" to Fig. 103, read $A_1F = E'A_2 = -1.5$.
P. 347. For "single spiherical surface" read "single spherical surface."
P. 558. In Fig. 167 insert letter V at point of intersection of straight lines MQ_1 and N_1D.
P. 597. Instead of heading "b, B" read "b."
An additional list is given at the end of this review.

H. Schulz and Max Lange, scientific collaborators of the optical firm of Goerz. Let us hope that, despite the cost of the original production and of subsequent revisions, a generous publisher or some appreciative optical firm in this country will see to it that the necessary financial arrangements may be made to keep the revision of our American edition up to the German translation. It would further be highly desirable to furnish the author with every possible encouragement to bring out a second volume on the special theory of optical instruments, the natural continuation of the present volume.

The first two chapters of our text are introductory, descriptive of the methods and fundamental laws of geometrical optics and of characteristic properties of rays of light. Huygens's construction as modified by Fresnel is made the basis. Mention is made of that important principle whereby the solution of a problem in reflection can be derived from the solution of the corresponding problem in refraction by taking -1 for the relative index of refraction. The principles of least time and shortest route, Malus's law and the characteristics of an infinitely narrow bundle of optical rays (due to Sturm) are explained.

Chapters III and IV deal with reflection and refraction at a plane surface and refraction through a prism or system of prisms. In the discussion of the deviation of a ray obliquely refracted through a prism, there is repeated a widespread error leading to the erroneous formula (57) which should read

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}E \cos \eta_1, \quad \text{not} \quad \cos \tfrac{1}{2}D = \cos \tfrac{1}{2}E \cos \eta_1.$$

This error has apparently had a long and honorable career. It occurs in the works of Heath, Czapski, Loewe, Kayser, and practically all others. It was detected and pointed out by Uhler;* it had previously been pointed out by Larmor in 1898; it was avoided and the correct result was obtained by Mascart in his Traité d'Optique (1889). It is unlikely that any large number of authors working the problem through independently should have come upon the same erroneous formula; but of course it is impossible for each author to work out each detail independently; his very learning the subject from an earlier trustworthy source produces an aberration toward the error. The eradication of the newer errors of science is constantly

* *American Journal of Science*, volume 27 (1909), pages 223–228.

going on, but the older errors are hard to dislodge; the long survival and widespread adoption of such a one as this points a moral not to the latest author alone but to the whole scientific world.

Chapters V and VI discuss refraction and reflection of paraxial rays* at a spherical surface and through a thin lens or system of thin lenses. It is here that the applications of projective geometry begin. This is carried on to the general treatment of the geometrical theory of optical imagery in Chapter VII. The theory is applied in the next chapter to lenses and lens systems. Thus approximately one half the text is used up in dealing with the simple elementary geometrical optics of prisms and paraxial systems of rays refracted (or reflected) at spherical surfaces.

Now although the paraxial ray is simple to handle, it is not sufficiently effective for the optician; for the wide-angle bundle of rays is a necessity both for the distinctness and the brightness of the image. The ninth chapter therefore takes up the exact methods of tracing the path of a ray refracted at a spherical surface; spherical aberration is defined, and the refraction formulæ of Kerber and Seidel are given. Next follow the forms for calculating the path of a ray through a centered system of spherical surfaces. The discussion then is carried up to the general case of the refraction of an infinitely narrow bundle of rays through an optical system, and the matter of astigmatism which had previously been treated for refraction at a plane is now expounded for the case of a spherical surface.

The next, the twelfth chapter, on the theory of spherical aberrations, is by far the longest in the book, as may easily be imagined from the nature of its subject. The author follows chiefly the method of Seidel, the elegance of which is largely due to a selection of line coordinates felicitous for the rays involved. Gauss wrote the equation of the ray in the familiar form

$$y = \frac{Bx}{n} + P, \quad z = \frac{Cx}{n} + Q,$$

and used the parameters B, C, P, Q (n is the index of refraction and the x-axis is the optical axis). Seidel used as ray

* A paraxial ray is one lying so near the axis that magnitudes of the second order relative to its deviation from the axis may be neglected.

coordinates the quantities $(\eta, \zeta, \eta, \zeta)$ where (η, ζ) and (η, ζ) are the coordinates of the two points in which the ray cuts two selected planes perpendicular to the x-axis. Especial attention is given to the terms of the 3d order, those of the 5th order being neglected.*

In Chapter XIII the question of color phenomena is treated, and the related subject of achromatism. The historical introduction and the description of what Jena glass has meant to optical engineering is an interesting relief from arrays of formulæ. The final chapter deals with the aperture and field of view, and the brightness of optical images. Here for the most part the mathematics is very simple. It is rare that the author applies physical reasoning instead of geometric, but a little is found in this last chapter and it recalls the fact that earlier he points out that Clausius from the second law of thermodynamics and Helmholtz from the law of conservation of energy could obtain the fundamental sine condition of Abbe.

It is clear that, from his consistent and logical presentation of the theory of geometrical optics, Professor Southall has attracted and must continue to attract highly favorable attention to himself, his institution, and his country. In this we should take pride. May he attract from us similar attention to his deserving subject.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

EDWIN BIDWELL WILSON.

As the above review was about to go to press, the BULLETIN received from Professor Southall the following additional list of errata in his Geometrical Optics, supplied by R. S. Clay, Esq., of London. It is believed that the publication of the full list will be of material service to the readers of the book.

Page 159. Insert a minus sign before the first term on the right-hand side of formula (81).

Page 237. Change x to x' in the right-hand part of Fig. 92.

Page 304. In the first line of formulæ (191), read $(\alpha - \alpha')$ instead of $(\alpha + \alpha')$.

Page 306. In the legend to Fig. 122, read $\angle CBG = \alpha$ instead of $\angle GBC = \alpha$.

Page 307. In formula (197) read $\angle BCA_g$ and $\angle BCA_i$ instead of the symbols ϕ_g and ϕ_i, respectively.

Page 310, near the bottom. Read $\angle BCA_g$ and $\angle BCA_i$ instead of ϕ_g and ϕ_i, respectively.

* More recently in the *Astrophysical Journal* (May, 1911), the author has taken up the subject again.

Page 365. In formulæ (268) read $f_u = \bar{f}_u \cdot \cos^2 \alpha = \text{etc.}$

Page 393. In the first term on the right-hand side of formula (298) in the denominator read $n_k - 1$ instead of n_{k-1}.

Page 414. In the second and third equations on this page, read $\Delta \frac{n \cdot \delta u}{uu}$ instead of $\Delta \frac{n \cdot \delta u}{u}$.

Page 414. In the fifth equation on this page, insert a minus sign before d_{k-1}.

Page 439. In the second term of formula (323), read $\bar{R}_m'$ instead of R_m'.

Page 461. In the third formula on this page, on the right-hand side, insert a multiplication-dot before the expression in brackets.

Page 462. At the top of this page, strike out the first line, and insert in place thereof the following:

If we neglect the terms of the 3d order, the direction-cosines of the incident ray may be regarded as 1, b/n, c/n; so that the approximate equations of the incident ray BH are:

Page 462. In the denominator of the fraction on the right-hand side of the equation in the 6th line read $n^2(y_k^2 + z_k^2)$ instead of $y_k^2 + z_k^2$.

Page 465. In the last expression in the last term of the first of equations (355) read J_k^3 instead of J_k.

Page 465. In the second of equations (355) in the first term on the right-hand side read $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{z}_1$ in place of $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{y}_1$; and after the + sign before the last term insert $\frac{1}{2}$.

Page 467. In the second of equations (357), in the numerator of the fraction in the first term on the right-hand side read $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{z}_1$ instead of $(\mathbf{y}_1^2 + \mathbf{z}_1^2)\mathbf{y}_1$.

Page 604. §122. Read Chap. I instead of Chap. II.

SHORTER NOTICES.

A Treatise on the Analytic Geometry of Three Dimensions. By GEORGE SALMON, late Provost of Trinity College, Dublin. Fifth edition, revised by R. A. P. ROGERS, Fellow of Trinity College, Dublin. Volume 1. London, Longmans, Green and Company, 1912. 8vo. xxii+470 pages, and two plates.

THE first edition of Salmon's Geometry of Three Dimensions was published in 1865. It formed the closing volume of an extensive treatise on algebraic geometry, two volumes of which were concerned with plane geometry, while the third contained a development and interpretation of the theory of linear transformations, from the standpoint of invariants, then just becoming known.

While many of the facts were known before, the point of view was a new one, and the great mass of material was

successfully moulded into a systematic treatment. Extensive use was made of various chapters of algebra, but the algebraic processes are always kept in the background, and a full explanation of the geometric meaning of every step was given.

Many of the foundations are not carefully laid, and a somewhat curious mixture of rigorous logic and of appeal to the intuition are found side by side. One great difficulty in the choice of material was happily overcome by the author by assuming that the reader was familiar with the contents of each of the earlier volumes of the treatise. In this way considerable use could be made of determinants, linear transformations, projective systems, and invariants by presupposing theorems stated elsewhere. A student who has approached the subject through these preparatory steps will have but little difficulty in following the argument.

These books have been translated into many of the languages of Europe, and have run through several editions. In the meantime more comprehensive treatises on the subject matter of any one of the parts taken separately have appeared, and new contributions have greatly extended the boundaries, yet for a general orientation into the field of analytic geometry these volumes have remained the standard work, and have also served as a model of style and of presentation to most writers on geometry. The second edition appeared in 1869, the main additions being a consideration of Clebsch's researches on algebraic curves, and of Cayley's contributions to the theory of ruled surfaces. The third edition followed five years later (1874), having been prepared with the assistance of Professor Cayley. Two innovations in this edition are the introduction of the Plücker line coordinates and the Gauss coordinates of points on a surface; the appendix on quaternions, which appeared in the preceding edition, was omitted. The fourth edition was issued in 1882, under the supervision of Mr. Cathcart. Professor Salmon had in the meantime almost entirely withdrawn from mathematics. Since 1866 he had been Regius professor of divinity in Trinity College, Dublin, and the duties of his new office left him less and less leisure to continue his earlier studies in geometry. The fourth edition contains fewer changes than the preceding ones, scarcely taking into account the advances that had been made on the continent during the eight years since the last preceding one was published.

Counting the translations, the volume on three dimensions has run through fourteen editions, the preparation of which has been participated in by a dozen different persons.

Under these circumstances, it is a significant fact that thirty years after the last English edition appeared, in a time of unprecedented activity in cultivating new territory in geometry, and of compiling systematic treatises on the same general plan, it is still felt that a new edition can be best presented by preserving all the characteristic features of the preceding ones, only adding such material as is necessary to have the development conform to the present point of view.

In November, 1910, the Board of Trinity College, Dublin, authorized the preparation of a fifth edition along these lines, and appointed Mr. Rogers, Fellow, to undertake the task. In order to accomplish this purpose the size of the large volume had to be still further increased, and it was decided to divide it into two. The present volume contains the first 12 chapters, and the number of pages of subject matter has increased from 382 to 455; it also contains two photographic plates of models of quadric surfaces.

The first two chapters contain no essential changes. The concept of surface is introduced as the totality of triads of numbers which satisfy one given equation. A curve is defined as the totality of triads satisfying two. It is stated that three surfaces meet in a finite number of points, no mention being made in this place of those surfaces which have a partial intersection in common, but attention is called to the fact that algebraic equations are to be understood.

The transition from rectangular to tetrahedral coordinates is still hopelessly abrupt; a few sentences inserted from time to time have made the new presentation more logical, but have scarcely lessened the difficulty.

An article is added at the end of Chapter V, expressing the analytic classification of quadrics in terms of invariants, but it is not complete; a number of assumptions are made, only part of which are explicitly mentioned.

In Chapter VII a few explanatory sentences are introduced, which make the discussion less abrupt; the idea of projectivity is made more prominent, and treated more systematically. At the end of the chapter two articles are added, one on projective coordinates, collineations, and reciprocation, and the other on the projective theory of distance and angle. In

the former, any tetrahedron and any unit point are shown to fix a system, and the coordinates are interpreted as anharmonic ratios, thus justifying the name projective coordinates. Projection is defined as a (1, 1) correspondence between the elements of any two entities, when the correspondence is defined by a system of linear equations with a non-vanishing determinant. In the second added article the projective theory of distance and angle are treated, thus justifying the earlier classification of quadrics. The discussion is also outlined for non-euclidean space, but the treatment is so brief that it can hardly be of assistance to the reader.

The chapter on confocal quadrics contains three added notes which connect the theory of confocal surfaces with that of poles and polars with regard to the absolute circle, and a number of proofs are simplified by means of duality. Use is made of the contents of these articles when discussing invariants and covariants of a system of quadrics. No other changes are made in this later chapter.

In the first part of the chapter on curves and developables, a considerable number of problems on curves defined by parametric equations, mostly concerning curves of order 3 or 4, is added. This excellent treatment of the projective properties of algebraic curves, the extension of Plücker's numbers to space curves, and the illustrations by means of curves of orders 3 and 4 is still standard, forty years after its first appearance.

In the second part of this chapter the Codazzi formulas and the Frenet formulas have been added, and Staude's construction of confocal quadrics is treated in fairly full outline.

This volume is provided with an index of subject matter and with a list of authors cited.

VIRGIL SNYDER.

Oeuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. III. Paris, Gauthier-Villars, 1912. 8vo. 522 pp.

A REVIEW of the first two volumes of Hermite's works has been given in volume 13, pages 182–190, and volume 16, pages 370–377 of this BULLETIN. The thirty-nine memoirs which make up the present volume belong to the years 1872–1880. A fourth volume will contain the remaining papers and bring the work to a close.

The majority of the papers of this third volume are short notes of a few pages each, and deal for the most part with algebra, the integral calculus, theory of numbers, differential equations, and elliptic functions. Three of these are extracts from Hermite's Cours d'Analyse; another is from his autographed lectures at the Ecole Polytechnique.

By far the most important paper is the celebrated "Applications des Fonctions elliptiques" appearing first in the *Comptes Rendus*, beginning in 1877, and afterwards in book form in 1885. This memoir occupies more than 150 pages of the present volume, and contains Hermite's epoch making researches on Lamé's differential equation.

The present volume brings another paper of less real importance but of far more sensational nature, namely the one on "La Fonction Exponentielle." Here Hermite shows that e, the base of the Naperian logarithms, is indeed a transcendental irrationality. A cruel fate robbed him of the glory of proving that π is also transcendental, and yet it would have been but a short step for him to make. In a letter to Borchardt he writes: "Je ne me hasarderai point à la recherche d'une démonstration de la transcendence du nombre π. Que d'autres tentent l'entreprise, nul ne sera plus heureux que moi de leur succès, mais croyez-m'en, mon cher ami, il ne laissera pas que de leur en côuter quelques efforts." Hermite had not long to wait, for nine years later, in 1882, Lindemann brought the long sought proof, and so established the impossibility of "squaring the circle."

The volume is graced with a portrait of Hermite, at about sixty-five. It is a striking likeness; but the kindly look about the eyes will be missed by those who knew him.

JAMES PIERPONT.

Naturwissenschaften und Mathematik im klassischen Altertum. Von J. L. HEIBERG, in Kopenhagen. Mit 2 Figuren im Text. Teubner, Leipzig, 1912. 102 pp. M. 1.25.

IN our generation there have been three men who were preeminently fitted both by taste and by training to write upon the mathematics of the classical civilization. Others have been able to undertake the task in a satisfactory manner, as witness the labors of scholars like Zeuthen, Loria, and Moritz Cantor, but there always stand out three names of men whose love for Greek science and perfect command of

the classical languages fitted them in a remarkable manner for investigation in this field. It is unnecessary to name these men to anyone who has worked in the history of mathematics, but it may be permitted, because of other readers, to mention the work of the late M. Paul Tannery, the great contributions of Sir Thomas Heath, and the noteworthy editions of the Greek classics in mathematics that have appeared from time to time under the editorship of Professor Heiberg.

It is therefore particularly fortunate that the publishers of "Aus Natur und Geisteswelt" were able to secure the services of Dr. Heiberg in the preparation of this little work. No one could speak with greater authority upon the subject, and few could so successfully condense the important facts for the general reader. Taken in connection with Professor Zeuthen's summary of the history of mathematics, now in proofsheets and soon to appear in Die Kultur der Gegenwart, the student of the subject will have two excellent points of departure for serious work.

Dr. Heiberg begins, as would of course be expected, with the natural philosophy of the Ionian school, covering a period in which mathematics, physics, and cosmography were closely allied. He then considers the Pythagorean movement, particularly with reference to astronomy, geometry, and the theory of numbers. A chapter is then assigned to medical science as it developed in the fifth century B. C., particularly under the influence of Hippocrates. The development of mathematics in the same period, from the time of Pythagoras to that of Plato, is then discussed. Chapters V and VI deal with the labors of Plato and of Aristotle, respectively, together with those of their followers. These are followed by the longest chapter in the work, one devoted to the Alexandrian period, in many respects the most important of all antiquity. Chapter VIII is happily entitled "Die Epigonenzeit," a period extending through the second and first centuries B. C. In this period we meet the names of Asclepiades of Bythnia in medicine, Alexandros of Myndos in zoology, Theodosius in the study of the sphere, Diocles in geometry, and various others who may properly be designated as epigones.

The last two chapters relate to the feeble contributions of Rome and to the Greek scientific literature of the Empire and the Byzantine period.

It is hardly necessary to remark that such a brief presentation of the subject cannot be critical in its nature. To some extent this defect is remedied by the brief bibliography at the end, and by a list of source material.

DAVID EUGENE SMITH.

Methodologisches und Philosophisches zur Elementar-Mathematik. Von G. MANNOURY. Haarlem, P. Visser Azn., 1909. 276 pp.

THERE appear from time to time, and in various countries, works of more or less merit that relate to the border line or the neutral ground between mathematics and philosophy, not attempting to eradicate existing boundaries, but seeking to show the relations that continually appear when one considers the two regions. We find the same thing on the other side of mathematics, where it borders upon the various physical sciences, and at the present time this region is particularly in the educational limelight. From the standpoint of the lover of pure science the former domain is the more interesting and important, while to him whose interests are chiefly in the utilities the latter has more significance.

Among the writers in our language who have of late contributed most successfully to the study of the borderland of philosophy and mathematics Bertrand Russell is perhaps the best known. In France M. Couturat has taken a prominent position, with the late lamented Poincaré writing with equal vigor in both regions. In Italy the writings of Peano, Pieri, and Veronese are well known, and other countries have contributed their quota to the study. It is, therefore, a helpful work that Dr. Mannoury has undertaken, to compile the views of various leading contributors to the study, while at the same time setting forth his own.

The work is divided into two parts, the first having to do with the foundations of arithmetic considered in its broadest sense, and the second with those of geometry. Under the former are considered in order the concepts of unity and multitude; of number, finiteness and infinity; of the distinctive fundamental principles of arithmetic; of the extension of the number concept and the principle of permanence; and of the irrational. As is often the case with continental writers the principle of permanence is attributed to Hankel, whereas Peacock introduced it in his Algebra nine years before Hankel

was born, and made much of it in his Treatise on Algebra in 1842. In the second part of the work the author begins with a dissertation on mathematical logic, a subject that has attracted so many writers in the last half century, and in the treatment of which the influence of Peano and Couturat is manifest. Chapter II treats of geometrography and the straight line, starting with Lemoine's work of twenty-five years ago and closing with the contributions of Poincaré and Russell. Chapter III develops the theory of non-euclidean geometry, and then shows the significance of various historical attempts to demonstrate the Euclid postulate or to found a geometry independent of this assumption. The work closes with a general discussion of the space concept.

Readers will find the chapters on the number concept and the non-euclidean geometry particularly interesting. As a genuine contribution to theory the book will be less regarded than as a résumé of the questions involved.

DAVID EUGENE SMITH.

Encyclopädie der Elementar-Mathematik. Von H. WEBER und J. WELLSTEIN. III Band: *Angewandte Mathematik.* Zweite Auflage. Erster Teil: *Mathematische Physik.* Bearbeitet von RUDOLF WEBER. Teubner, Leipzig und Berlin, 1910. 8vo. xiv+536 pages. 12 marks.

THE first edition of this encyclopedia was reviewed in the BULLETIN, volume 10 (1903–4), pages 200–204. In this second edition of the third volume, on applied mathematics, there are extensive changes. The original volume is divided into two; the present one, a treatise complete in itself on mathematical physics, and one to follow on graphics, probabilities, and astronomy. This modification has been made to satisfy many criticisms of the original, some of which deplored the wide omissions in a work that called itself an encyclopedia.

The present volume has three chapters on mechanics: functions of position and direction that appear in physics, analytic statics, and dynamics; two chapters on electric and magnetic fields: electricity and magnetism, and electromagnetism; two chapters on maxima and minima: geometric maxima and minima, and applications to the theories of equilibrium and of capillarity; two chapters on optics: geometric optics, and plane waves.

The first chapter is on vectors and has been completely rewritten. Two definitions of vector are given, one in substantially the usual terms, the other as follows: A vector $\mathfrak{A}$ is a function which depends not only on a located point, but also has a value for every direction radiating from the point. These values are its *components*, and must be such that any three of them which form a trirectangular system, when added geometrically, produce that one of the components which has the maximum length. This particular component however has no precedence over the others in importance and no one of the components is the vector, but the entire system of values correlated to the directions. Geometrically, the vector is represented by a pair of spheres tangent at the point in question, with the ensemble of chords drawn through the point, which chords are positive in one sphere and negative in the other. Examples of these "physical vectors" are found in forces, displacements, velocities, accelerations, electric and magnetic fields. This conception of vector as a set of function values, rather than as a directed line segment, or as a hypercomplex number, seems to us more like an attempt at novelty than at usefulness. The gain is not evident. In the text following, the analysis goes back practically to the usual mode of development. A section is added on "Tensoren," a name introduced by Voigt, which are the dyadics of Gibbs and the linear vector operators of Hamilton. Examples are the pressure in a deformed elastic body, elasticity coefficients, conductivities of heat, dielectric constants, and other properties of crystals.

The chapter on geometrical optics covers the usual ground. That on plane waves leads up to the electromagnetic theory of light. These extensive additions are an improvement in the original.

Many small changes have been made throughout, but we need not dwell on them. The criticisms made in the first review, referred to above, still hold in large measure.

JAMES BYRNIE SHAW.

Über freie und erzwungene Schwingungen. Von Dr. ARTHUR KORN. Teubner, Leipzig und Berlin, 1910. 8vo. vi+136 pages. M 5.60.

THE title of this memoir is somewhat deceptive, as it does not deal directly with the theory of oscillations, which are

mentioned only in a brief introduction of nine pages, where the solution of certain problems in oscillation is reduced to expansions in terms of normal functions (Eigenfunktionen) belonging to integral equations. The entire text is then taken up with the development of the theory of the solution of linear integral equations following the method of successive approximations. This method may, however, without too much strain on the sense of the terms, be called the method of development in series of oscillating functions, and possibly this idea may have suggested the title.

The treatment is divided into three sections, presenting respectively the theory of linear integral equations with continuous, symmetric kernel, with discontinuous symmetric kernel, and Fredholm's solution of linear integral equations with any continuous kernel. In an appendix some generalizations are given for kernels with discontinuities for many-dimensional problems, and for systems of linear integral equations. The book closes with a bibliography of memoirs by the author relating to the applications of the method of successive approximations. This method owes its development to C. Neumann, Poincaré, Picard, and Korn.*

In a sense the book is an introduction to the theory of developments in series of normal functions particularly when the kernel is unsymmetric. These are, up to the present time, special investigations which each extend the field a little. In many cases of particular forms of the kernel, it is possible to develop theorems analogous to those on which is based the solution of the case of a symmetric kernel. How far this method may lead one in such investigations remains for the future to show.

JAMES BYRNIE SHAW.

Etude sur l'Assurance complémentaire de l'Assurance sur la Vie. By P. J. RICHARD. Paris, A. Hermann et Fils, 1911. 118 pages.

THE appearance recently in this BULLETIN of the note that "at the University of Göttingen . . . candidates in applied mathematics must henceforth be prepared to be examined in the mathematics of insurance," the approval by the Italian Chamber of Deputies of the bill providing for a state monopoly

* H. Bateman, Report on the history and present state of the theory of integral equations, page 21.

of life insurance, and the recent British National Insurance Act, make more generally interesting the monograph before us.

The question is raised as to whether the insurance companies are going to be viewed sometime much as the express companies are to-day. When the ratio of management expenses (exclusive of taxes) to premium income for our American companies varies from 8.7 to 28.4, even to 227.6 for a state company, with an average of 22.4; when the company for which this ratio is below 20 is thought to be quite economically managed, one wonders whether the cost of insurance is not much too high. If the government had the monopoly of life insurance, as in some countries, would it not benefit our people as a parcels post does other people? If the company that has cut its cost of management to half of that of most of the best of other companies could flourish since 1759, and if the cost in Germany is still less, why should not our millions of people profit by what national control could save? The similarities between express and insurance companies cannot be considered here, neither can the causes of the great variation in cost of management.

The author of the study, in an introduction of six pages, states the object and aim of life insurance, calls attention to the possible failure to maintain it, and asks the question which has embarrassed many an agent: "If I fall ill and am no longer able to earn anything, who will pay my premiums?" This leads to the matter in hand, l'assurance complémentaire, which provides against the lapsing of a policy on account of temporary or permanent incapacity for labor due to accident or to disease. The premiums are guaranteed and the amount thus paid beforehand is deducted from the amount paid to the benefactor at the time of death of the insured. The history of this sort of insurance is briefly told. Reference is made to the fact that German and American companies sell such a contract. The regular life companies of France are warned by the Controle Central des Compagnies au Ministère du Travail not to enter into such contracts. The author states that the aim of the study is to show that this risk is as rightfully assurable as the risk of death. The companies have made a serious and rational study only as to equitable charges for the new combination. Though definite statistical data are lacking, the data furnished by other branches of insurance may be temporarily used to outline a rudimentary theory.

At present the companies are simply feeling their way to a reasonable tariff.

I have found about four American companies that have some form of select life policy, or clause in a regular policy, covering this field. It may be of considerable interest to many readers to see a sample clause of this sort:

"*Waiver of Premiums.*—The company, by endorsement hereon, will waive payment of the premiums thereafter becoming due, if the insured, before attaining the age of sixty years and after paying at least one full annual premium and before default in the payment of any subsequent premium, shall furnish proof satisfactory to the company that he has become wholly and permanently disabled by bodily injury or by disease so that he is and will be permanently, continuously and wholly prevented thereby from performing any work for compensation or profit, or from following any gainful occupation. Any premiums so waived shall not be deducted from the sum payable under the policy. Provided that, notwithstanding proof of disability may have been accepted by the company as satisfactory, the insured shall at any time, on demand, furnish the company satisfactory proof of the continuance of such disability; and if the insured shall fail to furnish such proof, or if it shall appear to the company that the insured is able to perform any work or to follow an occupation whatsoever for compensation, gain or profit, all premiums thereafter falling due must be paid in conformity with this contract.

"Without prejudice to any other cause of disability, the entire and irrecoverable loss of the sight of both eyes, or the severance of both hands above the wrists, or both feet above the ankles, or of one entire hand and one entire foot will be considered as total and permanent disability within the meaning of this provision."

The clear, concise statement characteristic of the whole treatment does much toward holding the reader's interest through the five chapters of the book, as the author treats minutely of the conditions of the complementary insurance and even as he calculates tables of morbidity, of healthy and invalid persons, of value of an indemnity for one and for two insured, of premiums for numerous particular forms of insurance, etc. Though with insufficient statistics the author seems to have presented a rational study of this interesting and very

little known form of insurance, pointing out, it would seem, every exigency that could possibly arise.

CHARLES C. GROVE.

Konstructionen und Approximationen. Von THEODOR VAHLEN. Teubner, Leipzig und Berlin, 1911. xii + 349 pp.

ONE who expects to find in this book—Band XXXIII of the Teubner Sammlung—a more or less complete list of constructions and approximations with a strong flavor of applied mathematics will be disappointed, as was the reviewer. According to the author it is intended to help bridge the gap which exists between the mathematics of the German gymnasium and university. The latter does not begin its work where the former ends. Various books, notably those by Klein and Enriques, have been published recently which might be studied by those intending to follow the lectures on higher mathematics. The book under review aims to furnish such preparation by having the student actually come in contact with some concrete facts in mathematics and to know these so well that later when the professor during his lecture has him soaring more or less he may still have a point or two of contact with the earth below.

The class of books having in view preparation for the university is decidedly different for Germany than for the United States. To illustrate this we might mention that the first 75 pages of the book under review are devoted to having the student obtain definite notions concerning the fundamental principles of projective geometry. Special emphasis is placed in all of its phases, both algebraic and geometric, on the interpretation of the cross-ratio. Good drill work, all of it. Of course, it couldn't be included in the lectures given later—that would seem too much like teaching.

Throughout the book the various aspects of the solutions of the three famous problems of antiquity are presented and many references to the literature on the subject given. Interesting metric cubic constructions in which algebra and geometry are closely correlated are cited. Approximate solutions of cubics and biquadratics are obtained geometrically and the limited range of constructions possible with ruler and compass pointed out. In this connection are included several solutions, ancient and modern, of the duplication of the cube and tri-

section of an angle and various pieces of apparatus designed to solve the same problems are described and their theory discussed. Later some interesting constructive approximations are given.

The properties of various transcendental curves are used to obtain approximately an nth root and to divide any angle into n equal parts. The division of the circle and of the arc of the lemniscate into n equal parts for special values of n with the aid of ruler and compass alone is discussed.

A development of attempts to arrive at the value of π from the time of Ahmes to that of Lindemann is presented. This leads naturally to mechanical quadrature and rectification.

Under the heading of analytic approximations are included such titles as Taylor's series, Lagrange's interpolation formula, exponential series with application to the quadrature of the hyperbola, De Moivre's theorem, indeterminate forms, and the determination of π by the use of series.

The discussion of the irrationality of π and π^2 brings out the methods used by famous mathematicians of old. The book closes with the proof of the transcendental nature of e and π.

There is much concrete work in algebra and geometry throughout the book, consequently a chance for errors, many of which have been listed in an appendix of two pages.

Student's mathematical clubs in our universities desiring some interesting material for the rounding out of a course in mathematics would find the volume rich in suggestions.

ERNEST W. PONZER.

NOTES.

THE sixth regular meeting of the Southwestern Section of the American Mathematical Society will be held at the University of Kansas on Saturday, November 30. Titles and abstracts of papers to be presented at this meeting should be in the hands of the chairman of the programme committee, Professor J. N. Van der Vries, University of Kansas, by November 8.

THE annual meeting of the Society will be held this year at Cleveland, Ohio, in affiliation with the American association

for the advancement of science, on Tuesday, Wednesday, and Thursday, December 31–January 2. The winter meeting of the Chicago Section will be merged with the annual meeting. Titles and abstracts of papers should be sent to the Secretary of the Society, 501 West 116th Street, New York, on or before December 10.

THE twentieth summer meeting of the Society will be held at the University of Wisconsin early in September, 1913. At the seventh colloquium of the Society, held in connection with this meeting, courses of lectures will be delivered as follows: By Professor W. F. OSGOOD: "Selected topics in the theory of analytic functions of several complex variables." By Professor L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of algebraic and modular invariants."

THE concluding (October) number of volume 34 of the *American Journal of Mathematics* contains: "Simple groups from order 2001 to order 3640," by L. P. SICELOFF; "On certain orthogonal systems of lines and the problem of determining surfaces referred to them," by A. E. YOUNG; "Wallace's theorem concerning plane polygons of the same area," by W. H. JACKSON; "On harmonic functions," by R. A. HARRIS; "Curves on quintic scrolls," by F. B. WILLIAMS; "On the structure of forms, and the algebraic theory of n-lines," by O. E. GLENN.

THE opening (September) number of volume 14 of the *Annals of Mathematics* contains the following papers: "Three-dimensional chains and the associated collineations in space," by HAZEL H. MACGREGOR; "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute," by E. J. MILES; "Theorems on reducible quantics," by O. E. GLENN; "A determinant formula for the number of ways of coloring a map," by G. D. BIRKHOFF.

AT the meeting of the Edinburgh Mathematical Society on June 14 the following papers were read: By Professor H. S. CARSLAW, "A problem in the linear flow of heat discussed from the point of view of the theory of integral equations"; by D. G. TAYLOR, "Linear substitutions and their invariants"; by WM. BRASH, "Two general results in the differential calculus."

THE International commission on the teaching of mathematics made its report at the Fifth international congress of mathematicians at Cambridge, England, in August. Reports were received from eighteen countries, and 150 separate reports were submitted. About fifty more are now in process of preparation, and others are contemplated by various countries. The central committee, consisting of Professor F. KLEIN (Göttingen), Sir GEORGE GREENHILL (London), and Professor H. FEHR (Geneva), with Professor DAVID EUGENE SMITH (New York) added, was continued in office for another period of four years. The American reports have been completed and may be obtained gratis by application to the Bureau of Education, Washington, D.C. It is probable that one or more reports, summarizing the larger features of the reports of all other countries, will be prepared by the American commission during the next four years, and that certain other special lines of work will be undertaken. The central committee contemplates holding three international conferences on teaching, the first in France in 1914, the second in Germany in 1915, and the third, with the next congress, in Stockholm in 1916.

THE fourteenth meeting of the Australasian association for the advancement of science will be held at Melbourne in January, 1913.

THE eighty-second annual meeting of the British association for the advancement of science was held at Dundee during the week from September 4–11 under the presidency of Professor E. A. SCHÄFER.

The meeting was divided into twelve sections, Section A, mathematics and physics, being under the chairmanship of Professor H. L. CALLENDAR. Programmes of the proposed proceedings for the day appeared each morning, together with the abstracts of the papers that were presented the preceding day, and the entertainments for each day, of which there was a very generous provision.

The following papers were presented before section A: Presidential address, "The nature of heat," by H. L. CALLENDAR; "The heating effect of radium emanation and its products," by E. RUTHERFORD and H. ROBINSON; "On the discharge of ultraviolet light of high-speed electrons," by R.

A. MILLIKAN; "Sur une nouvelle machine algébrique," by M. A. GERARDIN; "On Mersenne's numbers," by A. CUNNINGHAM; "On arithmetic factors of the Pellian terms," by A. CUNNINGHAM; "Atomic heat of solids," by F. A. LINDEMANN; "The algebraic numbers derived from the permutations of any assemblage of objects," by P. A. MACMAHON; "A mode of composition of positive quadratic forms," by E. H. MOORE; "Proof of a general theorem relating to orders of coincidence," by J. C. FIELDS; "The use of the exponential curve in graphics," by H. B. HEYWOOD; "Report on Bessel and other functions," by the committee appointed for that purpose. A number of other papers not of mathematical content were read before this section. In Section M, educational science, a symposium was devoted to the present position of mathematical teaching, in which papers were read by T. P. NUNN, P. PINKERTON, W. P. MILNE, and W. D. EGGAR.

The association has granted £30 to its committee, under the chairmanship of Professor M. J. M. HILL, for the continuation of the work of tabulating the Bessel functions.

The next meeting of the association will occur in Birmingham under the presidency of Sir W. H. WHITE.

THE one hundred and second edition of the complete catalogue of the publishing house of B. G. Teubner, Leipzig and Berlin, is dedicated to the Fifth international congress of mathematicians at Cambridge, and copies were presented to interested members. As frontispiece it contains a portrait of Euler, and three other photographic plates are contained in the volume; one shows the photographs of a number of prominent leaders in the organization of the Encyclopedia, another has a similar list from the "Kultur der Gegenwart," and a third from the workers for reform in the curricula of German schools.

The catalogue contains a history of the development of the house of Teubner, with particular mention of the two undertakings, the Encyclopedia of the mathematical sciences, and the publication of the works of Euler. About thirty books on mathematical subjects are mentioned as in press or in an advanced state of preparation.

THE following courses in mathematics are announced at the German universities during the winter semester, 1912–1913:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Differential calculus, four hours; Elliptic functions, four hours; Certain problems in maxima and minima treated by elementary geometry, two hours; Exercises in differential calculus, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Algebra, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: General theory of analytic functions, four hours; Potential function, four hours; Seminar, two hours.—By Professor G. HETTNER: Definite integrals, two hours.—By Professor J. KNOBLAUCH: Mathematical problems, four hours; Twisted curves and surfaces, four hours; Mathematical exercises, one hour.—By Professor R. LEHMANN-FILHÉS: Analytic geometry, four hours.—By Dr. K. KNOPP: Theory of numbers, four hours; Advanced function theory, four hours; Chapters from the theory of infinite series, one hour.

UNIVERSITY OF BRESLAU.—By Professor R. STURM: Plane analytic geometry, three hours; Theory of transformations, I, three hours.—By Professor A. KNESER: Exercises from the mathematico-physical seminar, two hours; Integral calculus, four hours; Theory of probability, two hours.—By Professor E. SCHMIDT: Theory of analytic functions, four hours; Theory of real functions, II, two hours.—By Dr. W. SCHNEE: Algebra, II, four hours; Exercises in mathematics, three hours.

UNIVERSITY OF ERLANGEN.—By Professor M. NOETHER: Differential and integral calculus, I, four hours; Theory of functions, four hours; Seminar.—By Professor E. FISCHER: Analytic geometry, I, four hours; Differential equations, three hours; Selected topics from advanced algebra, I, one hour.

UNIVERSITY OF GIESSEN.—By Professor L. SCHLESINGER: Theory of functions, three hours; Theory of numbers, two hours; Differential and integral calculus, four hours; with exercises, one hour; Seminar, one hour.—By Professor H. GRASSMANN: Plane analytic projective geometry, four hours; Descriptive geometry, II, five hours; Seminar, exercises on projective geometry, one hour.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Development of mathematics during the nineteenth century,

four hours; Seminar, two hours.—By Professor D. HILBERT: Theory of partial differential equations, two hours; Mathematical foundations of physics, two hours; Axioms of physics, Seminar, two hours.—By Professor C. RUNGE: Numerical calculation, with exercises, six hours; Selected chapters of mechanics, two hours.—By Professor E. LANDAU: Infinite series, four hours; Seminar, two hours.—By Professor F. BERNSTEIN: Mathematics of insurance, two hours.—By Dr. O. TOEPLITZ: Differential and integral calculus, II, four hours; Theory of invariants, two hours.—By Dr. H. WEYL: Theory of functions, four hours; Integral equations, three hours.—By Dr. T. v. KÁRMÁN: Mechanics, I, four hours.—By Dr. R. SCHIMMACK: Mathematical didactics, two hours.—By Dr. H. v. SANDEN: Descriptive geometry, four hours; with exercises, four hours.—By Dr. G. V. RÜMELIN: Introduction to the mathematical treatment of the natural sciences, three hours.—By Dr. R. COURANT: Determinants, four hours; Exercises in the theory of functions, two hours; Applications of determinants to geometry, four hours.—By Dr. P. HERTZ: Kinetic theory of gases, two hours; Elementary theory of numbers, four hours.

UNIVERSITY OF GREIFSWALD.—By Professor F. ENGEL: Differential geometry, four hours; Partial differential equations, four hours; Transformation groups, two hours; Seminar, two hours.—By Professor K. T. VAHLEN: Algebra, four hours; Statistics with emphasis on graphical methods, two hours; with exercises on graphical methods, two hours.—By Dr. W. BLASCHKE: Analytic geometry, five hours; Calculus of variations, two hours.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Ordinary differential equations, five hours.—By Professor L. HAUSSNER: Twisted curves and surfaces, four hours; Differential and integral calculus, with exercises, II, five hours; Analytic geometry of space, four hours; Proseminar, analytic geometry of space, two hours; Seminar, one hour.—By Professor G. FREGE: Analytic mechanics, four hours.—By Dr. M. WINKELMANN: Descriptive geometry, four hours; Exercises in descriptive geometry, two hours; Exercises in ordinary differential equations, one hour.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER:

Differential equations with one independent variable, four hours; Analytic geometry of space, four hours; Seminar, one hour.—By Professor G. LANDSBERG: Theory of numbers, four hours; Calculus of variations, four hours; Seminar, one hour.—By Professor M. DEHN: Integral calculus, four hours; Projective geometry, three hours; Exercises in applied mathematics, one hour.—By Dr. R. NEUENDORFF: Descriptive geometry, II, three hours.

UNIVERSITY OF MÜNSTER.—By Professor W. KILLING: Analytic geometry, II, four hours; Theory and applications of elliptic functions, two hours; Determinants and elementary algebra, two hours; Exercises in analytic geometry, one hour; Seminar, two hours.—By Professor R. v. LILIENTHAL: Differential and integral calculus, II, four hours; Theory of curvature of curves and surfaces, four hours; Political arithmetic and insurance, two hours; Exercises in integral calculus, one hour; Seminar, two hours.—By Dr. A. TIMPE: Synthetic geometry, three hours.

UNIVERSITY OF TÜBINGEN.—By A. v. BRILL: Introduction to higher mathematics, four hours; The mechanics of Hertz, three hours; Seminar, two hours.—By Professor L. MAURER: Elementary analysis, four hours; Integral equations, three hours; Seminar, two hours.—By Professor O. PERRON: Integral calculus, four hours; Theory of linear differential equations, three hours; Seminar, integral calculus, one hour.

UNIVERSITY OF PARIS. By Professor G. DARBOUX: Infinitesimal geometry and applications, two hours.—By Professor E. GOURSAT: Differential and integral calculus, elements of the theory of analytic functions, two hours.—By Professor E. BOREL: Functions of a large number of variables, one hour.—By Professor P. APPELL and M. C. GUICHARD: General laws of equilibrium and motion, two hours.—By Professor J. BOUSSINESQ: Mechanical properties of fluids and special types of fluid motion, two hours.—By Professor G. KOENIGS: Dynamics of continuous media from the point of view of the applications, two hours.—By MM. VESSIOT and MONTEL: Cours de mathématiques préparatoires, two hours.—By M. A. CAHEN: Fermat's last theorem, two hours.

Conferences will be conducted by MM. Lebesgue, Guichard, Vessiot, Montel, and Servant.

In the Ecole Normale courses in "mathematics" are given by Professors BOREL, CARTAN, and MM. VESSIOT and LEBESGUE.

In the second semester the following courses are announced: By Professor E. PICARD: Recent researches in the theory of analytic functions and in particular its relation to integral equations.—By Professor E. GOURSAT: Ordinary and partial differential equations.—By Professor P. APPELL: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics.—By Professor J. BOUSSINESQ: Waves of oscillation, emersion, and impulsion, sonorous waves.—By M. VESSIOT: Analysis and mechanics.—By M. A. CAHEN: Fermat's last theorem.

THE Academy dei Lincei at Rome has received from Dr. GINO MODIGLIANI, the sum of 4000*l*, to be used in the publication of the works of Leonardo da Vinci.

PROFESSOR G. LORIA, of the University of Genoa, and Professor R. MARCOLONGO, of the University of Naples, have been elected corresponding members of the Academy dei Lincei at Rome.

AT the United States Naval Academy, Professor H. E. SMITH has been appointed head of the department of mathematics and mechanics. The former ranking members of the department, Professors S. J. BROWN and H. M. PAUL, have been assigned to duty elsewhere.

AT the Georgia School of Technology, Professor FLOYD FIELD, head of the department of mathematics, has been granted a year's leave of absence for study at Harvard University. Mr. W. V. SKILES has been appointed associate professor of mathematics and acting head of the department during Professor Field's absence. Mr. L. W. MURPHY has been appointed assistant professor of mathematics.

PROFESSOR S. L. BOOTHROYD, of Cornell University, has been appointed associate professor of mathematics and astronomy at the University of Washington.

PROFESSOR G. P. PAINE, of the University of Minnesota, has been appointed assistant professor of mathematics at Middlebury College.

At Yale University Dr. W. A. Wilson has been promoted to an assistant professorship of mathematics.

At the University of Wisconsin Dr. Arnold Dresden has been promoted to an assistant professorship of mathematics.

At the University of North Dakota Mr. R. R. Hitchcock has been promoted to an assistant professorship of mathematics.

Professor W. P. Russell, of Pomona College, has been promoted to an associate professorship of mathematics.

Professor C. S. Atchinson, of Williams College, has been appointed professor of pure mathematics in Washington and Jefferson College.

Dr. S. A. Urner, of Miami University, has been promoted to an assistant professorship of mathematics.

At Hamilton College Professor W. M. Carruth has been promoted to an associate professorship of mathematics.

Dr. S. D. Killam has been appointed instructor in mathematics in the University of Rochester.

Mr. Cornelius Gouwens has been appointed instructor in mathematics at the University of Iowa.

Professor R. E. Allardice, of Stanford University, has been granted leave of absence for the present academic year.

Dr. G. E. Wahlin, of the University of Illinois, has been granted leave of absence during the entire academic year and will study in Europe.

Professor C. H. Beckett, of Purdue University, has resigned to engage in actuarial work.

The death is announced of Lucien Lévy, past president of the mathematical society of France, at the age of 59 years.

Professor F. Kotter, of the Berlin technical high school, died August 17, at the age of sixty-one years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALDUS (R.). Zur Theorie der gegenseitig mehrdeutigen algebraischen Ebenentransformationen. (Habilitationsschrift.) Erlangen, 1911. 8vo. 36 pp.

BAUM (K. T.). Der Kreis und seine Quadrate, die arithmetische Wage, $\pi = 3{,}125$ und die Quadratur des Kreises. 2 Teile. Saarbrücken, Schmidtke, 1912. 8vo. 32 pp. M. 1.75

BOCHOW (K.). Der Kreis als Maximalfläche. Die wichtigsten Fälle des isoperimetrischen Problems für ebene Figuren. (Progr.) Nordhausen, 1912. 4to. 32 pp.

BÖHM (F.). Ueber die Transformation von homogenen bilinearen Differentialausdrücken. (Habilitationsschrift.) München, 1911. 8vo. 34 pp.

BÜCHEL (C.). Die Arithmetica des Diophant von Alexandria. (Progr.) Hamburg, 1912. 8vo. 38 pp.

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome II, volume 5, fascicule 1: Equations et opérations fonctionnelles, par S. Pincherle. Interpolation trigonométrique, par H. Burckhardt et E. Esclangon. Fonctions sphériques, par A. Wangerin et A. Lambert. Leipzig, Teubner, 1912. 8vo. 160 pp. M. 6.00

—— des sciences mathématiques. Edition française. Tome IV, volume 2, fascicule 1: Fondements géométriques de la statique, par H. E. Timerding et L. Lévy. Géométrie des masses, par G. Jung et E. Carvallo. Cinématique, par A. Schoenflies et G. Koenigs. Leipzig, Teubner, 1912. 8vo. 224 pp. M. 8.40

ENRIQUES (F.). See QUESTIONI.

ENZYKLOPÄDIE der mathematischen Wissenschaften. Band VI, 2, Lieferung 2: Bahnbestimmung der Doppelsterne und Satelliten, von J. v. Hepperger. Prinzipien der Störungstheorie und allgemeine Theorie der Bahnkurven in dynamischen Problemen, von E. T. Whittaker. Leipzig, Teubner, 1912. 8vo. Pp. 463–556. M. 3.00

FAINBERG (M.). Bestimmung einer Minimalfläche, deren Begrenzung aus zwei in parallelen Ebenen gelegenen geradlinigen Winkeln besteht. (Diss.) Strassburg, 1912. 8vo. 19 pp.

FALCKENBERG. Verzweigungen von Lösungen nichtlinearer Differentialgleichungen. (Diss.) Erlangen, 1912. 8vo. 32 pp.

FAZZARI (G.). See FREYCINET (C. de).

FERRAI (F.). Untersuchungen über das Striktionssystem einer einfach unendlichen Flächenschar nebst Anwendung auf das Hauptachsen-Problem der Flächen zweiter Ordnung. (Progr.) Kottbus, 1912. 8vo. 31 pp.

FREYCINET (C. de). Dell'esperienza in geometria. Traduzione di G. Fazzari. Palermo, Reber, 1912. 16mo. 16+119 pp. L. 2.00

FRICKE (R.) und KLEIN (F.). Vorlesungen über die Theorie der automorphen Funktionen. Band II: Die funktionentheoretischen Ausführungen und die Anwendungen. 3te Lieferung: Direkte Beweismethoden der Fundamentaltheoreme und Anhang. Leipzig, Teubner, 1912. 8vo. Pp. 439–668. M. 11.00

GÜSSMAR (J.). Lineare Gleichungen zwischen parabolischen Koordinaten. (Diss.) Rostock, 1911. 8vo. 92 pp.

HAEUSSLER (J. W.). Beweis des Fermatschen Satzes. Berlin, Krayn, 1912. 8vo. 48 pp. M. 1.50

HAIMERL (F. X.). Der grosse Fermatsche Satz. Nürnberg, Korn, 1912. 8vo. 16 pp. M. 0.60

HECHT (B.). Ueber rationale Dreiecke. (Progr.) Königsberg, 1912. 4to. 7 pp.

HEIN (R.). Die verschiedenen Arten der Symmetrie. (Progr.) Linz, 1911. 8vo. 21 pp.

HEIN (W.). Konstruktion von Oskulationsflächen zweiten Grades an krumme Flächen. (Progr.) Wien, 1912. 8vo. 16 pp.

HOFFMANN (T.). Ueber fehlerhafte mathematische Ausdrucksweisen. (Progr.) Zwickau, 1912. 4to. 29 pp.

HUBER (E.). Kombinationen zu bestimmten Summen. (Progr.) Amberg, 1912. 8vo. 23 pp.

IBRÜGGER (C.). Ueber die Grundlagen der Geometrie. 1ter Teil: Die geometrischen Axiome im Urteil des Rationalismus und Empirismus. (Progr.) Stargard, 1912. 8vo. 42 pp.

JONES (A. C.). An introduction to algebraical geometry. London, Frowde, 1912. 8vo. 548 pp. 12 s.

KLEIN (F.). See FRICKE (R.).

KNOTT (C. G.). Life and scientific work of Peter Guthrie Tait; supplementing the two volumes of scientific papers published in 1898 and 1900. New York, Putnam, 1911. 4to. 9+379 pp. Cloth. $3.25

KÖSSLER (H.). Ueber windschiefe Kegelschnitte. (Diss.) Halle, 1911. 8vo. 66 pp.

LATTES (G.). Contributo allo studio della flessione dei cilindri. (Dissertazione di laurea.) Torino, Gili, 1912. 8vo. 55 pp.

LEVI (F.). Integritätsbereiche und Körper dritten Grades. (Diss.) Strassburg, 1911. 8vo. 30 pp.

LUNGER (E.). Konstruktion des Rückkehrkurvenpunktes auf einer erzeugenden einer abwickelbaren Fläche. (Progr.) Dornbirn, 1912. 8vo. 6 pp.

MEYER (S.). Struktureigenschaften der projektiven Invarianten von Formen mit n Variabeln. (Diss.) Strassburg, 1911. 8vo. 43 pp.

MIDOLO (P.). Archimede e il suo tempo. Siracusa, Tamburo, 1912. 8vo. 25+523 pp. L. 11.00

MOHR (R.). Die Bertrandschen Kurven in der Theorie der Normalensysteme. (Diss.) Strassburg, 1911. 8vo. 38 pp.

MUSSOTTER (R.). Eine Skizze der Geschichte der Infinitesimalrechnung. (Progr.) Wien, 1912. 8vo. 21 pp.

MUTH (F.). Ueber solche Koordinatensysteme auf Flächen, bei denen die eine Schar von Parameterkurven auf der anderen gleiche Stücke abschneidet. (Diss.) München, 1912. 8vo. 54 pp.

OVIDIO (E. D'). Geometria analitica. 4a edizione, riveduta e corretta. (Biblioteca matematica, volume III.) Torino, Bocca, 1912. 8vo. 16+520 pp. L. 14.00

PAYER (K.). Ueber Konstruktionen der Kegelschnitte aus gegebenen Bestimmungsstücken. (Progr.) Troppau, 1912. 8vo. 28 pp.

PERNA (A.). Di alcune notevoli disposizioni e dell' Hessiano di un produtto di fattori lineari ennarii. Napoli, L. Pierro, 1912. 8vo. 21 pp.

PIERPONT (J.). Lectures on the theory of functions of real variables. Volume II. Boston, Ginn, 1912. 8vo. 13+645 pp.

POMAROLI (G.). Analytische Darstellung der Durchmesser von Raumkurven 3. Ordnung. (Progr.) Villach, 1912. 8vo. 26 pp.

QUESTIONI riguardanti le matematiche elementari, raccolte e coordinate di F. Enriques. Volume I: Critica dei principî. Articoli di U. Amaldi, R. Bonola, F. Enriques, D. Gigli, A. Guarducci, G. Vailati, G. Vitali. Bologna, Zanichelli, 1912. 8vo. 6+649 pp. L. 20.00

REICHEL (W.). Der mathematische Gedächtnisstoff. Leipzig, Quelle und Meyer, 1912. 8vo. 43 pp. M. 0.80

RUDOLPHI (W.). Analytische Geometrie des Raumes in Verbindung mit darstellender Geometrie. 2ter Teil: Die Kugel. (Progr.) Neumünster, 1912. 4to. 20 pp.

SCHALLER (J. G.). Beweis der Richtigkeit des grossen Fermat'schen Satzes. Grabow, 1912. 8vo. 23 pp. M. 0.75

SUINI (A.). Dissertazione sopra la possibilità di più geometrie fra loro antagonistiche, ecc. Piacenza, Porta, 1912. 8vo. 23 pp.

THEISS (F.). Der grosse Fermatsche Lehrsatz, entwickelt und dargestellt. Mainz, Diemer, 1912. 8vo. 61 pp. M. 3.00

TIEDEMANN (K.). Zur Theorie der Elimination. (Diss.) Königsberg, 1912. 8vo. 61 pp.

TOSCANO (S. A.). Sul calcolo dei momenti, in relazione alle operazioni formali dell' algebra vettoriale. Noto, Zammit, 1911. 8vo. 21 pp.

TOSCHI DI FAGNANO (G. C. DE). Opere matematiche. Volume III: Altri scritti scientifici, scritti polemici, carteggio, biografia. Roma, Bertero, 1912. 8vo. 11+227 pp.

II. ELEMENTARY MATHEMATICS.

BARONI (E.). Algebra. Volume I. 3a edizione. Firenze, Bemporad, 1912. 16mo. 6+212 pp. L. 1.80

BASSANI (A.). See LAZZERI (G.).

BETZ (W.) und WEBB (H. E.). Plane geometry. With the editorial cooperation of P. F. Smith. Boston, Ginn, 1912. 12mo. 10+332 pp. Cloth. $1.00

BLAINE (R. G.). Some quick and easy methods of calculating. 4th edition. London, Spon, 1912. 12mo. 164 pp. 2s. 6d.

BORCHARDT (W. G.) and PERROTT (A. D.). Geometry for schools. Volumes 1 to 4. London, Bell, 1912, 8vo. 3s-

BREWSTER (G. W.). See SANDERSON (F. W.).

FELDMAN (D. F.). See HART (C. A.).

GAUSS (F. G.). Fünfstellige vollständige trigonometrische und polygonometrische Tafeln. 2te Auflage. Stuttgart, Wittwer, 1912. 8vo. 100+18 pp. Cloth. M. 7.00

HART (C. A.) and FELDMAN (D. F.). Plane and solid geometry; with the editorial cooperation of J. H. Tanner and V. Snyder. New York, American Book Co., 1912. 12mo. 7+488 pp. $1.25

JONES (H. S.). Exercises in modern arithmetic. With answers. London, Macmillan, 1912. 8vo. 346 pp. 2s. 6d.

LAYNG (A. E.). Elementary geometry. London, Murray, 1912. 8vo. 264 pp. 3s. 6d.

LAZZERI (G.) e BASSANI (A.). Primi elementi di geometria. 2a edizione riveduta. Livorno, Giusti, 1912. 16mo. 7+101 pp. L. 1.40

LENNES (N. J.). See SLAUGHT (H. E.).

MATRICULATION model answers, mathematics. (University tutorial series.) London, Clive, 1912. 8vo. Sewed. 2s.

—— model answers, mechanics. London, Clive, 1912. 8vo. Sewed. 2s.

OXFORD local examinations. Papers of the examination held in July, 1912, with the answers to the questions set in mathematics and physics. Regulations for 1912. London, J. Parker, 1912. 8vo. Sewed. 2s.

PERROTT (A. D.). See BORCHARDT (W. G.).

PRICE (E. A.). Examples in numerical trigonometry. Cambridge, University Press, 1912. 8vo. 100 pp. 2s.

RAGANTI (B.). Aritmetica fondamentale. Sarzano, Rolla-Canale, 1911. 16mo. 11+335 pp. L. 3.50

RUBINSTEIN (A.). Plane geometry. New York, Hinds, Noble and Eldredge, 1912. 12mo. 5+200 pp. Cloth. $1.00

SANDERSON (F. W.) and BREWSTER (G. W.). Examples from A Geometry for Schools, with answers. Cambridge, University Press, 1912. 8vo. 158 pp. 1s. 6d.

SLAUGHT (H. E.) and LENNES (N. J.). First principles of algebra. Complete course. Boston, Allyn & Bacon, 1912. 12mo. 10+481 pp. $1.20

SOCCI (A.) e TOLOMEI (G.). Elementi di geometria secondo il metodo di Euclide. Volume I. 19a edizione. Firenze, Le Monnier, 1912. 8vo. 22+141 pp. L. 1.75

——. Elementi di matematica. Volume III. Firenze, Le Monnier, 1912. 8vo. 6+194 pp. L. 1.50

STAINER (W. J.). Graphs in arithmetic, algebra and trigonometry. London, Mills & Boon, 1912. 8vo. 228 pp. 2s. 6d.

TABLES of logarithms, anti-logarithms and reciprocals. London, Layton, 1912. 8vo. 1s.

TAYLOR (E. O.). An introduction to geometry. London, Clarendon Press, 1912. 8vo. 1s. 6d.

TOLOMEI (G.). See SOCCI (A.).

UNTERRICHT, Der mathematische, in der Schweiz. Berichte. Nr. 3, von S. E. Gubler, E. R. Scherrer, und K. Matter; Nr. 5, von L. Crelier; Nr. 6, von L. Morf. Basel, Georg, 1912. 8vo. 109+112+70 pp. M. 6.50

WEBB (H. E.). See BETZ (W.).

III. APPLIED MATHEMATICS.

ADLER (A. A.). The principles of parallel projecting-line drawing. New York, Van Nostrand, 1912. 8vo. 6+66 pp. $1.00

AKESSON (O. A.). Ueber eine symmetrische Form der analytischen Lösung des Bahnbestimmungsproblems. Berlin, 1911. 8vo. 24 pp.

BLAUERT (M.). Ueber einige Anwendungen der elliptischen Funktionen auf die Theorie des ebenen Gelenkvierecks. (Diss.) Rostock, 1911. 8vo. 123 pp.

BOHLIN (K.). Integralentwicklungen des von Haerdtl'schen Dreikörper-Problems. Berlin, 1911. 8vo. 36 pp.

FARROW (F. W.). Stresses and strains. 2d edition. revised. London, Whittaker, 1912. 8vo. 150 pp. 5s.

GRASSI (U.). Preparazione matematica allo studio della chimica fisica. Firenze, Le Monnier, 1912. 8vo. 4+248 pp. L. 4.00

GROSSMANN (M.). Einführung in die darstellende Geometrie. 2te, neu bearbeitete Auflage. Basel, Helbing und Lichtenhahn, 1912. 8vo. 4+92+12 pp. Cloth. M. 2.80

GRÜTTNER (A.). Die Grundlagen der Geometrographie. Leipzig, Quelle und Meyer, 1912. 8vo. 54 pp. M. 0.80

JUDE (R. H.) and SATTERLY (J.). Junior magnetism and electricity. London, Clive, 1912. 8vo. 296 pp. 2s. 6d.

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